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On certain mathematical aspects of nonlinear acoustics: well-posedness, interface coupling, and shape optimization

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Abstract

The research presented in this thesis is motivated by many applications of high intensity focused ultrasound, ranging from kidney stone treatments, welding, and heat therapy to sonochemistry, where a better understanding of the physical effects through mathematical analysis and optimization is expected to lead to improvement in precision and reduction of risks.

A significant part of the thesis is dedicated to the question of well-posedness for the Westervelt equation with strong nonlinear damping of $q$-Laplace type under practically relevant Neumann as well as absorbing boundary conditions. We prove local in time well-posedness through a fixed point approach under the assumption that the initial and boundary data are sufficiently small. We also obtain short time well-posedness for the problem modeling the interface coupling of acoustic regions, since this is a common occurrence in applications, e.g. lithotripsy.

Secondly, higher interior regularity for the model in question, as well as for the coupled system, is achieved by employing the difference quotient approach. We also show that the result can be extended up to the boundary of the subdomains, i.e. the solution to the coupled problem exhibits piecewise $H^2$-regularity in space, provided that the gradient of the acoustic pressure is essentially bounded in space and time on the whole domain. This result is of importance in future numerical approximations of the present problem, as well as in shape optimization problems governed by this model.

The last part of the thesis is dedicated to the shape sensitivity analysis for an optimization problem arising in lithotripsy. The goal is to find the optimal shape of a focusing acoustic lens so that the desired acoustic pressure at a kidney stone is achieved. We follow the variational approach to calculating the shape derivative of the cost functional which does not require computing the shape derivative of the state variable; however, assumptions of certain spatial regularity of the primal and the adjoint state are needed to obtain the derivative, in particular to express it in its strong form in terms of boundary integrals.
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CHAPTER 1

Introduction

Research on nonlinear acoustics has become an increasingly active field in recent years due to many medical and industrial applications of high intensity focused ultrasound. In lithotripsy, ultrasound is used for nonsurgical treatment of hardened masses in the body such as kidney stones and gallstones. Focused ultrasound can also be used as a form of heat therapy, where it generates highly localized heating to treat cysts and benign or malignant tumors (see, e.g., [65, 66]). Since ultrasound can have an effect of forming acoustic cavitation in liquids, it is used in sonochemistry to initiate and enhance chemical activity in solutions. Ultrasonic welding is a popular industrial technique for joining dissimilar materials by applying high-frequency ultrasonic acoustic vibrations to objects which are held together under pressure. As a cleaning technique, application of ultrasound allows for milder chemicals to be used in the cleaning process in comparison to other cleaning techniques. Other applications of focused ultrasound include cataract treatments, noninvasive treatment of certain brain disorders, bone and wound healing, cosmetic medicine, etc. A detailed overview of possible ultrasound applications can be found, for instance, in [19].

Due to the nonlinear effects observed in the propagation of ultrasound in these cases, such as the appearance of sawtooth solutions, studying models of nonlinear acoustics is of great importance. Although there exists a number of publications on the physical modeling (see, e.g., [12, 13, 26, 32, 42, 49, 58, 63]) and on simulation of high intensity ultrasound (see, e.g., [18, 31, 40, 67]) in physics and engineering literature, rigorous mathematical treatment of the equations of nonlinear acoustics is still a fairly recent and incomplete undertaking.

![fig. 1.1: Schematic of a power source in lithotripsy based on the electromagnetic principle](image-url)
Chapter 1. Introduction

The work in this thesis is motivated, e.g., by lithotripsy, where mathematical analysis, numerical simulation, and optimization are expected to lead to a better understanding and control of the physical effects and thus to a substantial reduction of lesions and a decrease in complication risks during treatment.

The design of currently used devices in lithotripsy is mainly based on two different principles (see, e.g., [38, 39, 40, 61]): excitation and self-focusing by a piezo mosaic or an electro-magneto-mechanical principle. In the first case, a piezoelectric array of transducers is arranged on the inner surface of a spherical calotte (see fig. 1.2) and each element is aimed at the focal point. In the latter case, Lorentz forces acting on a membrane radiate an acoustic pulse in a fluid (see fig. 2.1); the pulse is then focused by a silicone lens at a kidney stone. Our research will be focused on the case where the lens is modeled as an acoustic medium surrounded by a nonlinearly acoustic fluid. This raises an important issue, namely

♦ modeling and analysis of the coupling of (nonlinear) acoustic regions, under suitable interface and boundary conditions.

In view of this, a considerable part of the present work is devoted to the questions of well-posedness and regularity for the governing PDE models under Dirichlet, or practically more relevant Neumann and absorbing boundary conditions.

![fig. 1.2: Schematic of a HIFU device with piezoelectric excitation](image)

In construction of lithotripters based on the electromagnetic principle, naturally the question arises of

♦ optimal design of the lens focusing the ultrasound at the kidney stone.

So far high intensity ultrasound devices have been designed by employing numerical simulations of the underlying partial differential equation models on account of engineering experience without the use of mathematical optimization methods. For a recent result we refer to [50].

However, shape optimization and shape sensitivity analysis represent a sound mathematical foundation for implementing efficient and reliable optimization tools, which would here support and improve the already existing engineering approaches. Having this in mind, a significant
part of the present work is dedicated to the mathematically rigorous derivation of first order
sensitivities in the context of the respective governing PDE model, which will serve as a basis
for developing an effective optimization algorithm.

Contents of the chapters

Chapter 2 introduces the most popular model in nonlinear acoustics - Westervelt’s
equation, as well as the model we will be working with which has an added strong nonlinear
damping term of $q$-Laplace type. We will present the governing equations and lay out the
inherent difficulties of working with this model.

Chapter 3 contains an overview of certain well-known theoretical results concerning
Sobolev spaces and ODE existence theory as well as a number of helpful inequalities, which
are all often employed in the rest of the thesis.

Chapter 4 is dedicated to the question of well-posedness for the Westervelt equation with
strong nonlinear damping. Unlike [9], we investigate this model under Neumann as well as
absorbing boundary conditions. Additionally, advantages of introducing lower order linear and
nonlinear damping terms are investigated. Moreover, well-posedness in case of acoustic-acoustic
coupling, as relevant in devices following schematic from figure 1.1, is shown. The results
presented here are based on the following work:

- V. Nikolić, Local existence results for the Westervelt equation with nonlinear damping and
  Neumann as well as absorbing boundary conditions, Journal of Mathematical Analysis

Chapter 5 deals with the question of higher interior regularity for the Westervelt equation
with strong nonlinear damping as well as for the system of these models coupled through
interface conditions. Regularity up to the boundary/interface is also obtained under the
assumption that the gradient of the acoustic pressure stays essentially bounded in time and
space; this represents the foundation for the shape sensitivity analysis in the next chapter.
Chapter 5 is based on the following work:

- V. Nikolić and B. Kaltenbacher, On higher regularity for the Westervelt equation with

Chapter 6 provides shape sensitivity analysis for the problem of finding the optimal shape
of a focusing lens in lithotripsy (cf. fig. 1.1) and gives an answer to the question of existence of
such optimal shapes. Shape derivative is obtained through a variational approach introduced
by Ito, Kunisch and Peichl in [30], which does not require shape differentiability of the state
variable. This chapter relies on the following result:

- V. Nikolić and B. Kaltenbacher, Sensitivity analysis for shape optimization of a focusing

Comment on the notation. Throughout the thesis, we will employ the dot notation for
differentiation (also known as Newton’s notation) with respect to the time variable, i.e. $\dot{u}$ and
$\ddot{u}$ will denote the first and the second time derivative of a function $u$. 
CHAPTER 2

Westervelt’s equation

In this chapter, we will present the most commonly used mathematical model in nonlinear acoustics - Westervelt’s equation. It is named after the American physicist Peter Westervelt (1919–2015), renowned for, among other things, introducing the concept of the parametric array in [63]. Westervelt’s equation originated as a by-product of this discovery.

2.1 Governing equations

The propagation of waves is described by the following quantities (see, for example, [39, Chapter 5]):

- time and spatial variation of the density \( \rho_\sim = \frac{\text{mass}}{\text{volume}} \, [\text{kg/m}^3] \),
- pressure \( u_\sim = \frac{\text{force}}{\text{cross section}} \, [\text{N/m}^2] \),
- velocity \( v_\sim = \frac{\text{distance}}{\text{time}} \, [\text{m/s}] \).

These quantities can be decomposed into their mean (whose time and - in case of homogeneous fluids - also space derivatives vanish) and alternating part as follows:

\[ \rho_\sim = \rho_0 + \rho, \quad u_\sim = u_0 + u, \quad v_\sim = v_0 + v. \]

Here \( \rho \) denotes the acoustic density and \( u \) the acoustic pressure, and \( v \) is the acoustic particle velocity. For the derivation of the Westervelt equation, zero mean velocity is assumed, i.e. \( v_\sim = v \). The acoustic field in a fluid can be fully described by the following equations:

- Navier-Stokes equation for momentum conservation:

\[ \rho(\dot{v} + (v \cdot \nabla)v) + \nabla u = \mu_V \Delta v + \left( \frac{\mu_V}{3} + \eta_V \right) \nabla (\nabla \cdot v), \]

here \( \mu_V \) denotes the shear viscosity and \( \eta_V \) the bulk viscosity; under the assumption that \( \nabla \times v = 0 \) the equation reduces to

\[ \rho(\dot{v} + (v \cdot \nabla)v) + \nabla u = \left( \frac{4\mu_V}{3} + \eta_V \right) \Delta v; \]

- the equation of continuity (conservation of mass):

\[ \nabla \cdot (\rho v) = -\dot{\rho}; \]
Chapter 2. Westervelt’s equation

- the state equation, which relates acoustic pressure $u$ and density $\varrho$ within the fluid:

\[ \varrho = \frac{u}{c^2} - \frac{1}{\varrho_0 c^4} \frac{B}{2A} u^2 - \frac{\kappa}{\varrho_0 c^4} \left( \frac{1}{c_V} - \frac{1}{c_u} \right) \dot{u}, \]

where $\kappa$ denotes the adiabatic exponent and $c_V$, $c_u$ the specific heat capacitance at constant volume and constant pressure, respectively, and $B/A$ represents the parameter of nonlinearity.

By merging the three equations into a single wave equation and replacing any physical quantity in a second-order term by its linearization, we arrive at the Kuznetsov equation

\[ \Delta u - \frac{1}{c^2} \ddot{u} = -\frac{b}{c^2} \Delta \dot{u} - \frac{1}{\varrho_0 c^4} \frac{B}{2A} \frac{d^2}{dt^2} u^2 - \frac{\varrho_0}{c^2} \frac{d^2}{dt^2} (\mathbf{v} \cdot \mathbf{v}), \]

where $b$ is the diffusivity of the sound

\[ b = \frac{1}{\varrho_0} \left( \frac{4\mu V}{3} + \eta V \right) + \frac{\kappa}{\varrho_0} \left( \frac{1}{c_V} - \frac{1}{c_u} \right). \]

If we ignore local nonlinear effects which are represented by the quadratic velocity term, we obtain the Westervelt equation:

\[ \Delta u - \frac{1}{c^2} \ddot{u} = -\frac{b}{c^2} \Delta \dot{u} - \frac{1}{\varrho_0 c^4} \frac{B}{2A} \frac{d^2}{dt^2} u^2, \]

which can also be written as

\[ (1 - 2ku)\ddot{u} - c^2 \Delta u - b \Delta \dot{u} = 2k(\dot{u})^2, \]

where $k = \beta_a/\lambda$, $\lambda = \varrho c^2$ is the bulk modulus, and $\beta_a = 1 + B/(2A)$. For a detailed derivation of (2.1) we refer to [26, 39, 63].

Westervelt’s equation can also be rewritten in terms of the acoustic velocity potential $\psi$:

\[ (1 - 2\tilde{k}\dot{\psi})\ddot{\psi} - c^2 \Delta \psi - b \Delta \dot{\psi} = 0, \]

with $\varrho \dot{\psi} = u$ and $\tilde{k} = k \varrho$.

2.1.1 Mathematical point of view. Westervelt’s equation is a quasilinear wave equation with a strong linear damping. When dealing with this model from a mathematical standpoint, special attention has to be given to the fact that it can degenerate if the factor $1 - 2ku$ is equal to zero. This means that any analysis of this equation has to include bounding away from zero this term, which boils down to finding an essential bound for the acoustic pressure $u$.

The first mathematically rigorous treatment of the Westervelt equation was performed in [24] by Kaltenbacher and Lasiecka. There Westervelt’s equation was considered with homogeneous Dirichlet boundary conditions and global well-posedness and exponential decay rates for the energy were obtained. The degeneracy of the Westervelt equation was avoided by making use of the Sobolev embedding

\[ H^2(\Omega) \hookrightarrow L^\infty(\Omega), \ \Omega \subset \mathbb{R}^d, \ d \in \{1, 2, 3\}, \]

in combination with a bound on $\Delta u$ obtained through energy estimates. In [39], global well-posedness was also shown for the case of inhomogeneous Dirichlet boundary conditions.
and in [35] the results were extended to the Neumann problem for the Westervelt equation; in both cases the equation was kept away from degenerating by employing (2.3).

Due to this need to steer clear of degeneracy, for the Westervelt equation only solutions which are very smooth, i.e. $H^2$-regular in space, can be shown to exist. However, since one of our central tasks is to study the interface coupling of acoustic regions, which is a common occurrence in applications, achieving this regularity over the whole domain will be too high of a demand. As we will see, these acoustic regions will possess different material parameters and the normal derivative of the acoustic pressure will jump over the joint interface, thus making $H^2$-regularity of the acoustic pressure over the whole domain impossible.

### 2.2 Westervelt’s equation with strong nonlinear damping

As a way of relaxing regularity of solutions, and having acoustic-acoustic coupling in mind, Westervelt’s equation is considered with an added strong nonlinear damping term:

\[
(1 - 2ku)\ddot{u} - c^2 \Delta u - \text{div}(b((1 - \delta) + \delta|\nabla \dot{u}|^{q-1})\nabla \dot{u}) = 2k(\dot{u})^2
\]

with $\delta \in (0, 1)$, $q \geq 1$, $q > d - 1$, $d \in \{1, 2, 3\}$. For this model, which was first introduced in [9], degeneracy can be avoided by the means of the embedding

\[
W^{1,q+1}(\Omega) \hookrightarrow L^\infty(\Omega), \quad \Omega \subset \mathbb{R}^d, \quad d \in \{1, 2, 3\}, \quad q > d - 1,
\]

in combination with a bound on $|u(t)|_{W^{1,q+1}(\Omega)}$, which can be obtained through energy estimates. Using this model therefore allows to show existence of $W^{1,q+1}$-regular in space weak solutions, and in turn well-posedness of the acoustic-acoustic coupling problem.

Introduction of the $q$-Laplace term to the Westervelt equation is, on the other hand, motivated by models for power-law fluids (see, e.g., [62]), where the effective viscosity is proportional to certain power of the shear rate, which is used in the relation

\[
\text{shear stress} = \text{shear rate times effective viscosity}.
\]

The shear rate is always a spatial derivative of the velocity, which in its turn (via the linearized version of the Navier-Stokes equation) can be rewritten in terms of the pressure gradient. This is a physical motivation behind the appearance of the gradient of pressure under the $q$-power in the model.

#### 2.2.1 $q$-Laplacian

In comparison to [21], equation (2.4) has a new important feature, namely a damping term obtained by applying the $q$-Laplace operator

\[
\Delta_q u := -\text{div}(|\nabla u|^{q-1}\nabla u),
\]

$q \geq 1$, to the first time derivative of $u$:

\[
\Delta_q \dot{u} := -\text{div}(|\nabla \dot{u}|^{q-1}\nabla \dot{u}).
\]

In the special case of $q = 1$, (2.5) reduces to the standard Laplacian.

Working with the new model [24] implies handling this $q$-Laplace damping term in terms of analysis, numerics and optimization, which is a nontrivial task. The $q$-Laplace equation, given by

\[
-\Delta_q u = f,
\]
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and the parabolic $q$-Laplace equation

$$\dot{u} - \Delta_q u = f,$$

(2.7)

where $f$ belongs to an appropriate function space, have been extensively studied in the past with respect to regularity (see, for example, [46, 60, 16, 21] and references given therein) and numerical treatment (see, for instance, [17, 6, 47, 7]).

Regarding shape optimization problems which are governed by $q$-Laplace type equations, the few existing results seem to trace back to [10], where the continuity of the solution to a Dirichlet problem was considered with respect to domain perturbations. For a recent account, we refer to [11], where the optimal placement of a Dirichlet region in a domain, governed by (2.6), was studied. However, investigations on the interface coupling of $q$-Laplace type equations, in terms of numerics and optimization, seem to be lacking in the mathematical literature.

Results on hyperbolic equations with damping of $q$-Laplace type are sparse and so far have been mostly concerned with local and global well-posedness. In [64, 54] a wave equation with a $q$-Laplace damping and a nonlinear source was considered and global well-posedness was shown. Global existence and asymptotic behavior were achieved in [25] for a nonlinear wave equation with a $q$-Laplace damping term. However, other features of (2.4), such as the potential degeneracy, were not present in [25] and [64, 54].

In [9], equation (2.4) was investigated for the first time, under homogeneous Dirichlet boundary conditions and local in time well-posedness was obtained, as well as the short time well-posedness for an interface coupling of these models under homogeneous Dirichlet boundary conditions on the outer boundary.

A shape optimization problem governed by the Westervelt equation (2.1) was recently considered in [37]; however, there are seemingly no considerations in literature of shape optimization problems governed by hyperbolic equations with $q$-Laplace damping terms or coupled systems of these types of models (save for our work in [53] which will be presented here).

2.2.2 Different nonlinear damping terms. In [9, 51], the possibility of adding different nonlinear damping terms to the Westervelt equation (2.1) was also investigated, namely adding a $q$-Laplace damping term

$$(1 - 2ku)\ddot{u} - c^2\Delta u - \text{div}(b((1 - \delta) + \delta|\nabla u|^{q-1})\nabla u) = 2k(\dot{u})^2,$$

(2.8)

as well as

$$\ddot{\psi} - \frac{c^2}{1 - 2k\psi} \Delta u - b\text{div}(((1 - \delta) + \delta|\nabla \dot{\psi}|^{q-1})\nabla \dot{\psi}) = 0;$$

(2.9)

the second model is motivated by the acoustic velocity potential formulation (2.2). However, for these models only local existence of solutions could be shown, but not well-posedness, mainly because it was not possible to obtain higher order energy estimates. In this way, model (2.4) turned out to be the most “well-behaved”. For more details on (2.8) and (2.9), we refer to [9, Section 3 and 6] and [51, Section 4 and 5].

2.3 Interface coupling in nonlinear acoustics

Coupling of acoustic regions over a joint interface is an important occurrence in high intensity focused ultrasound applications, e.g. in lithotripsy, where an acoustic lens, which
2.3. Interface coupling in nonlinear acoustics

focuses the ultrasound, is immersed in an acoustic fluid medium (see fig. 1.1). Naturally, the question of modeling and well-posedness in this context arises.

The case of the acoustic-acoustic coupling can be modeled by the presence of spatially varying coefficients, with possible jumps over interfaces between subdomains, in the weak form of the equation (2.4) (see [5] for the linear, and [9] and [51] for the nonlinear case):

\[
\begin{aligned}
\text{Find } u & \text{ such that } \\
\int_0^T \int_\Omega \left\{ \frac{1}{\lambda(x)}(1 - 2k(x)u)\ddot{u}\phi + \frac{1}{\varrho(x)}\nabla u \cdot \nabla \phi + b(x)(1 - \delta(x))\nabla \dot{u} \cdot \nabla \phi \\
+ b(x)\delta(x)|\nabla \dot{u}|^{q-1}\nabla \dot{u} \cdot \nabla \phi - \frac{2k(x)}{\lambda(x)}(\ddot{u})^2\phi \right\} \, dx \, ds = 0
\end{aligned}
\]

holds for all test functions \( \phi \in \tilde{X} \), with \( (u, \dot{u})_{t=0} = (u_0, u_1) \), and appropriately chosen test space \( \tilde{X} \) (which will be specified in the forthcoming chapters together with assumptions on the coefficients). In this model \( b \) denotes the quotient between the diffusivity and the bulk modulus, while other coefficients retain their meaning. For notational brevity, we emphasized the space dependence of coefficients in (2.10), while omitting space and time dependence of \( u \) and the test function in the notation.

In the next chapter, we will tackle the question of local in time well-posedness for this model, and later on it will also appear as a governing model when optimizing the shape of the focusing lens.

We also briefly mention that the focusing lens in lithotripsy can be modeled as an elastic medium, which leads to the problem of an elastic-acoustic coupling. This more advanced model, was considered in [9]:

\[
(2.11) \quad \varphi(x)\ddot{\psi} - B^T \frac{1}{1 - 2k(x)\varphi} [c](x)B\psi + B^T \left( ((1 - \delta(x)) + \delta(x)|B\psi|^{q-1}|b|(x)B\psi \right) = 0,
\]

under homogeneous Dirichlet boundary conditions and suitable assumptions on the spatially varying coefficients. Here \( \varphi \) denotes the acoustic velocity potential, \( \phi \) stands for the gradient part in the Helmholtz decomposition of \( \psi \), \( \psi = \nabla \phi + \nabla \times A \), and the first order differential operator \( B \) is given by

\[
B = \begin{pmatrix}
\partial_{x_1} & 0 & 0 & 0 & \partial_{x_3} & \partial_{x_2} \\
0 & \partial_{x_2} & 0 & \partial_{x_3} & 0 & \partial_{x_1} \\
0 & 0 & \partial_{x_3} & \partial_{x_2} & \partial_{x_1} & 0
\end{pmatrix}^T.
\]

For this model local in time well-posedness was obtained in [9] under Dirichlet boundary conditions.
CHAPTER 3

Theoretical preliminaries

In this chapter, we will collect certain well-known theoretical results which will be often employed throughout the rest of the thesis as well as a number of helpful inequalities.

3.1 Sets of classes $C^l$ and $C^{l,1}$

Going forward, we will be working with either Lipschitz domains or at several instances $C^{1,1}$-regular domains. We recall their definition here:

**Definition 3.1.** [23, Definition 1.2.1.1] Let $l,m \in \mathbb{N}$, $1 \leq l,m \leq \infty$ and let $\Omega$ be an open subset of $\mathbb{R}^d$. We say that its boundary $\Gamma$ is continuous (respectively Lipschitz, continuously differentiable, of class $C^{l,1}$, $m$ times continuously differentiable) if for every $x \in \Gamma$ there exists a neighborhood $U$ of $x$ in $\mathbb{R}^d$ and new orthogonal coordinates $(y_1, \ldots, y_d)$ such that

(i) $U$ is a hypercube in the new coordinates:

$$U = \{(y_1, \ldots, y_d) : -a_j < y_j < a_j, \ 1 \leq j \leq d\};$$

(ii) there exists a continuous (respectively Lipschitz, continuously differentiable, of class $C^{l,1}$, $m$ times continuously differentiable) function $\varphi$, defined in

$$U' = \{(y_1, \ldots, y_{d-1}) : -a_j < y_j < a_j, \ 1 \leq j \leq d-1\},$$

and such that

$$|\varphi(y')| \leq \frac{a_d}{2}, \text{ for every } y' = (y_1, \ldots, y_{d-1}) \in U',$$

$$\Omega \cap U = \{y = (y', y_d) \in U : y_d < \varphi(y')\},$$

$$\Gamma \cap U = \{y = (y', y_d) \in U : y_d = \varphi(y')\}.$$

A $C^{l,1}$-mapping with a $C^{l,1}$-inverse is called a $C^{l,1}$-diffeomorphism.

**Theorem 3.2.** [14, Theorem 2.6] Let $\Omega$ be a Lipschitz domain in $\mathbb{R}^d$. Then for all integers $l \geq 0$, $W^{l+1,\infty}(\Omega) = C^{l,1}(\Omega)$ algebraically and topologically.

3.2 Sobolev embeddings

In the remaining of the thesis, we will employ a number of Sobolev embeddings, which we for that reason recall here. For a thorough introduction and results on Sobolev spaces we refer the reader to [1].

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DEFINITION 3.3. [4], Definition 7.3.6 Let $X$ and $Y$ be two Banach spaces over the domain $\Omega$ with $X \subset Y$. We say that the space $X$ is continuously embedded in $Y$ and write $X \hookrightarrow Y$ if 
(3.1) 
$$
\|u\|_X \leq C_{X,Y}^d \|u\|_Y, \ \forall u \in X.
$$

The space $X$ is compactly embedded in $Y$, denoted $X \hookrightarrow \hookrightarrow Y$, if (3.1) holds and each bounded sequence in $X$ has a convergent subsequence in $Y$.

THEOREM 3.4. [4], Theorem 7.3.7 Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. The following continuous embeddings hold:
- if $l < \frac{d}{r}$, then $W^{l,r}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \leq r^*$, $\frac{1}{r^*} = \frac{1}{r} - \frac{1}{d}$,
- if $l = \frac{d}{r}$, then $W^{l,r}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q < \infty$,
- if $l > \frac{d}{r}$, then $W^{l,r}(\Omega) \hookrightarrow C^{l-[\frac{d}{r}]_1,\beta}(\Omega)$, where 
$$
\beta = \begin{cases} 
\left\lfloor \frac{d}{r} \right\rfloor + 1 - \frac{d}{r}, & \text{if } \frac{d}{r} \neq \text{integer}, \\
\text{any positive number } < 1, & \text{if } \frac{d}{r} = \text{integer}.
\end{cases}
$$

THEOREM 3.5. [4], Theorem 7.3.8 Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. The following compact embeddings hold:
- if $l < \frac{d}{r}$, then $W^{l,r}(\Omega) \hookrightarrow \hookrightarrow L^q(\Omega)$ for any $q < r^*$, $\frac{1}{r^*} = \frac{1}{r} - \frac{1}{d}$,
- if $l = \frac{d}{r}$, then $W^{l,r}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q < \infty$,
- if $l > \frac{d}{r}$, then $W^{l,r}(\Omega) \hookrightarrow \hookrightarrow C^{l-[\frac{d}{r}]_1,\beta}(\Omega)$, where $\beta \in [0, \left\lfloor \frac{d}{r} \right\rfloor + 1 - \frac{d}{r})$.

In particular, the following embeddings hold and will be often employed throughout the thesis:
(3.2) $H^1(\Omega) \hookrightarrow \hookrightarrow L^4(\Omega)$, with the norm $C_{H^1,L4}^\Omega$,
(3.3) $L^r(\Omega) \hookrightarrow L^q(\Omega), \ r \geq q$, with the norm $C_{L^r,L^q}^{\Omega}$,
(3.4) $W^{1,q+1}(\Omega) \hookrightarrow L^\infty(\Omega), \ q > d - 1$, with the norm $C_{W^{1,q+1},L\infty}^{\Omega}$.

3.3 Traces

We also briefly recall the results on the density of smooth functions, as well as the concept of traces in Sobolev spaces. For an in-depth survey of this topic, we refer to [1] and [14].

THEOREM 3.6. (Meyers-Serrin, [14], Theorem 10.15) Let $\Omega \subset \mathbb{R}^d$ be an open set and let $1 \leq r < \infty$. Then the space $C^\infty(\Omega) \cap W^{1,r}(\Omega)$ is dense in $W^{1,r}(\Omega)$.

Note that the Meyers-Serrin theorem does not hold for functions in $W^{1,\infty}(\Omega)$.

THEOREM 3.7. [23], Theorem 1.5.1.3 Let $\Omega$ be a bounded open subset of $\mathbb{R}^d$ with a Lipschitz boundary $\Gamma$. Then the mapping 
$$
\gamma \mapsto u|\gamma,
$$
which is defined for $u \in C(\overline{\Omega})$, has a unique continuous extension as an operator from $W^{1,r}(\Omega)$ onto $W^{1-\frac{1}{r}}(\Gamma), \ 1 < r < \infty$. This operator has a right continuous inverse which does not depend on $r$. 

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The extended operator is called the trace operator and will be in future denoted by $\text{tr}_\Gamma$.

**Theorem 3.8.** [23, Theorem 1.5.1.2] Let $\Omega$ be a bounded open subset of $\mathbb{R}^d$ with a $C^{1,1}$ boundary $\Gamma$. Assume that $l - \frac{1}{r}$ is not an integer, $1 < r < \infty$, $l \leq 2$, $l - \frac{1}{r} = l_0 + \sigma$, $0 < \sigma < 1$, and $l_0$ is a nonnegative integer. Then the mapping $u \mapsto \frac{\partial u}{\partial n}|_\Gamma$, which is defined for $u \in C^1(\Omega)$, has a unique continuous extension as an operator from $W^{l,r}(\Omega)$ onto $W^{l-1,\frac{1}{r}}(\Gamma)$. This operator has a right continuous inverse which does not depend on $r$.

**Theorem 3.9.** [15, Theorem 3.42] Let $\Omega$ be a class $C^1$ open set and let $u \in W^{1,r}(\Omega)$. Then there exists a sequence $\{u_n\} \subset C^\infty(\Omega) \cap W^{1,r}(\Omega)$ that converges to $u$ in $W^{1,r}(\Omega)$ and satisfies $\text{tr}_\Gamma u_n = \text{tr}_\Gamma u$.

### 3.4 Fundamental theorems

Let us collect here three fundamental theorems which we will reference in the forthcoming chapters.

**Theorem 3.10.** (Dominated convergence theorem, [20, Theorem 4, Appendix E]) Assume the functions $\{f_m\}_{m=1}^\infty$ are integrable and $f_m \rightarrow f$ a.e. Suppose also that $|f_m| \leq g$ a.e. for some summable function $g$. Then

$$\int_{\mathbb{R}^d} f_m \, dx \rightarrow \int_{\mathbb{R}^d} f \, dx.$$ 

**Theorem 3.11.** (Banach’s fixed-point theorem, [55, Theorem 1.12]) A contractive mapping $T : X \rightarrow X$ on a Banach space $X$ has a unique fixed point $u$, i.e. $T(u) = u$.

**Theorem 3.12.** (Bolzano-Weierstrass, [55, Theorem 1.8]) Every lower (respectively upper) semicontinuous function $X \rightarrow \mathbb{R}$ on a compact set attains its minimum (respectively maximum) on this set.

### 3.5 A local existence result for ordinary differential equations

As an auxiliary problem in Chapter 4 we will encounter an initial-value problem of the following type:

\begin{align}
\dot{u} &= f(t, u(t)) \text{ for a.e. } t \in [0, T] \\
\quad \quad \quad u(0) &= u_0,
\end{align}

where the right hand side $f : (0, +\infty) \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ will be a Carathéodory mapping:

**Definition 3.13.** [55, Chapter 1] Considering integers $j, m_0, \ldots, m_j$ we say that a mapping $a : (0, T) \times \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_j} \rightarrow \mathbb{R}^{m_0}$ is a Carathéodory mapping if $a(\cdot, r_1, \ldots, r_j) : (0, T) \rightarrow \mathbb{R}^{m_0}$ is measurable for all $(r_1, \ldots, r_j) \in \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_j}$ and $a(t, \cdot) : \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_j} \rightarrow \mathbb{R}^{m_0}$ is continuous for a.e. $t \in (0, T)$.

By solution on a time interval $[0, T]$ of (3.5) we will call an absolutely continuous mapping $u : [0, T] \rightarrow \mathbb{R}^k$ such that the equation in (3.5) holds a.e. on $[0, T]$ and $u(0) = u_0$. The following short-time existence result holds:

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Theorem 3.14. [55, Theorem 1.44] Let \( f : (0, +\infty) \times \mathbb{R}^k \to \mathbb{R}^k \) be a Carathéodory mapping. Then there exists \( T > 0 \) such that the initial-value problem (3.5) has a solution on the interval \([0, T]\).

3.6 Helpful inequalities

Finally, we gather here essential inequalities which we will rely upon throughout the rest of the thesis.

In the case of problems with inhomogeneous Neumann boundary data it is often necessary to employ Poincaré’s inequality valid for functions in \( W^{1,q+1}(\Omega) \). We recall such inequality (see, e.g., [44, Theorem 12.23]), namely that there exists a constant \( C_P > 0 \) depending on \( q \) and \( \Omega \) such that

\[
|\varphi - \frac{1}{|\Omega|} \int_{\Omega} \varphi \, dx|_{L^{q+1}(\Omega)} \leq C_P |\nabla \varphi|_{L^{q+1}(\Omega)},
\]

for all \( \varphi \in W^{1,q+1}(\Omega) \).

Throughout the thesis, we will assume that \( t \in [0, T] \), where \( T \) is a finite time horizon. From (3.6) we can obtain

\[
|u(t)|_{W^{1,q+1}(\Omega)} \leq (1 + C_P) |\nabla u(t)|_{L^{q+1}(\Omega)} + C_1^Q \left| \int_{\Omega} u(t) \, dx \right|,
\]

and by replacing \( u \) with \( \dot{u} \) also

\[
|\dot{u}(t)|_{W^{1,q+1}(\Omega)} \leq (1 + C_P) |\nabla \dot{u}(t)|_{L^{q+1}(\Omega)} + C_2^Q |\dot{u}(t)|_{L^2(\Omega)},
\]
a.e. in time, where \( C_1^Q = |\Omega|^{-\frac{q}{q+1}} \) and \( C_2^Q = |\Omega|^{-\frac{q-1}{q+1}} \). By employing the embedding (3.4) and estimate (3.7), we can as well obtain

\[
|u(t)|_{L^\infty(\Omega)} \leq C_1^Q |u(t)|_{W^{1,q+1}(\Omega)} \leq C_1^Q \left( 1 + C_P |\nabla u(t)|_{L^{q+1}(\Omega)} + C_1^Q |\int_{\Omega} u(t) \, dx| \right) \\
\leq C_1^Q \left( 1 + C_P |\nabla u_0|_{L^{q+1}(\Omega)} + \int_0^t |\nabla \dot{u}(s)|_{L^{q+1}(\Omega)} \left| \int_{\Omega} u(t) \, dx \right| ds \right) \\
\leq C_1^Q \left( 1 + C_P |\nabla u_0|_{L^{q+1}(\Omega)} + t \int_0^t |\nabla \dot{u}|_{L^{q+1}(\Omega)}^{q+1} ds \right) \\
+ C_2^Q \left| \int_{\Omega} u(t) \, dx \right|_{L^2(\Omega)} \\
\leq C_1^Q |u_0|_{L^1(\Omega)} + C_2^Q \int_0^t |\dot{u}(t)|_{L^2(\Omega)} \, ds,
\]

which leads to the estimate

\[
\|u\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C_1^Q \left( 1 + C_P |\nabla u_0|_{L^{q+1}(\Omega)} + T \int_0^T |\nabla \dot{u}|_{L^{q+1}(\Omega)} \, ds \right) \\
+ C_1^Q |u_0|_{L^1(\Omega)} + C_2^Q T \|\dot{u}\|_{L^\infty(0,T;L^2(\Omega))},
\]

that will be employed when dealing with the possible degeneracy of the factor \( 1 - 2ku \). To obtain these estimates, we have assumed that \( u \) is sufficiently regular in time and space so that
3.6. Helpful inequalities

the right hand sides make sense.
We will also frequently make use of Young’s inequality in the ε-form
\[(3.10)\]
\[xy \leq \varepsilon x^{r} + C(\varepsilon,r) y^{\frac{1}{r-1}} \quad (x, y > 0, \varepsilon > 0, 1 < r < \infty),\]
with \(C(\varepsilon,r) = (r-1)r^{\frac{1}{r-1}} \varepsilon^{-\frac{r}{r-1}}.\)

### 3.6.1 Inequalities involving \(q\)-Laplacian.
Let us also recall several useful inequalities that we will need when handling the \(q\)-Laplace damping term. They can be found in [46, Chapter 10] and [47, Appendix]. From now on, \(C_q\) will be used to denote a generic constant depending only on \(q\). For any \(x, y \in \mathbb{R}^d\) and \(\eta \geq 0\) it holds
\[(3.11)\]
\[||x||^{q-1}_q x - ||y||^{q-1}_q y \leq C_q ||x - y||^{1-q} q \eta ||x||^{q-1+\eta}, q > 0,\]
\[(3.12)\]
\[(||x||^{q-1}_q x - ||y||^{q-1}_q y) \cdot (x - y) \geq 2^{1-q} ||x||^{q-1} q \eta ||x||^{q-1} + ||y||^{q-1} q \eta \geq 0, q \geq 1,\]
\[(3.13)\]
\[\frac{4}{(q+1)^2} ||x||^{\frac{q-1}{q+1}} q x - ||y||^{\frac{q-1}{q+1}} q y \leq ((||x||^{q-1}_q x - ||y||^{q-1}_q y) \cdot (x - y), q \geq 1,\]
\[(3.14)\]
\[||x||^{q-1}_q x - ||y||^{q-1}_q y \leq q(||x||^{\frac{q-1}{q+1}} q + ||y||^{\frac{q-1}{q+1}} q)||x||^{\frac{q-1}{q+1}} q x - ||y||^{\frac{q-1}{q+1}} q y, q \geq 1.\]

From (3.11) we also get
\[(3.15)\]
\[||x||^{q'} - ||y||^{q'} \leq C_q ||x - y||^{1-q} q \eta q ||x||^{q-1+\eta} + ||y||^{q-1+\eta},\]
for \(0 \leq \eta \leq 1, q > 1.\)

In several instances, we will utilize the following representation formula for vectors \(x, y \in \mathbb{R}^d\) (cf. [46, Chapter 10]):
\[(3.16)\]
\[\frac{|x|^{q-1}_q x - |y|^{q-1}_q y}{(x - y)} = (x - y) \int_0^1 |y + \sigma(x - y)|^{q-1}_q d\sigma + (q - 1) \int_0^1 \mathcal{L}(y + \sigma(x - y), (x - y)) d\sigma,\]
where
\[(3.17)\]
\[\mathcal{L}(x, y) = |x|^{q-3}_q (x \cdot y) x\]
as well as the inequality
\[|y + \sigma(x - y)|^{q-1}_q \leq |y|^{q-1}_q + |x|^{q-1}_q, \sigma \in [0, 1].\]

With the notation (3.17), as a simple consequence of (3.11), we have for vectors \(x, y, z, w\) and any \(\eta \geq 0, q > 2\) that
\[(3.18)\]
\[\mathcal{L}(x, y) = \mathcal{L}(z, w)\]
\[\leq C_q |x - z|^{1-\eta}(|x|^{q-3+\eta} + |y|^{q-3+\eta}) |x||y| + |z|^{q-2}(|y - w||x| + |w||x - z|).\]
CHAPTER 4

Local well-posedness results for the Westervelt equation with nonlinear damping

In this chapter, we will investigate the Westervelt equation with strong nonlinear damping \(2.4\), under practically relevant absorbing and Neumann boundary conditions. Our research into this area is motivated by many applications of high intensity focused ultrasound where the need for realistic boundary conditions is evident. Typically in acoustics one faces the problem of a physically unbounded domain which should then be truncated for numerical computations. Absorbing boundary conditions are used as a way of avoiding reflections on the artificial boundary \(\hat{\Gamma}\) of the computational domain. Ultrasound excitation, e.g. by piezoelectric transducers (see fig. 1.2), can be modeled by Neumann boundary conditions on the rest of the boundary \(\Gamma = \partial \Omega \setminus \hat{\Gamma}\). At the end of this chapter we will also consider the acoustic-acoustic coupling arising when focusing is done by an acoustic lens immersed in a fluid, cf. fig. 1.1.

4.1 Dirichlet boundary conditions

Before we proceed to the case of Neumann and absorbing boundary conditions, let us recall the local well-posedness result for the Dirichlet problem, obtained by Brunnhuber, Kaltenbach, and Radu in \(9\). We assume \(\Omega \subset \mathbb{R}^d\), \(d \in \{1, 2, 3\}\) to be an open, connected, bounded set with Lipschitz boundary and consider the following problem:

\[
\begin{aligned}
(1 - 2k u) \ddot{u} - c^2 \Delta u - b \text{div} \left( \left( (1 - \delta) + \delta |\nabla \dot{u}|^{q-1} \right) \nabla \dot{u} \right) &= 2k \dot{u}^2 \quad \text{in } \Omega \times (0, T], \\
u &= 0 \quad \text{on } \partial \Omega \times (0, T], \\
(u, \dot{u}) &= (u_0, u_1) \quad \text{on } \overline{\Omega} \times \{t = 0\},
\end{aligned}
\]

(4.1)

with the following assumptions on the coefficients and the exponent \(q\):

\[
c^2, \ b > 0, \ \delta \in (0, 1), \ k \in \mathbb{R}, \ q > d - 1, \ q \geq 1.
\]

(4.2)

The weak formulation reads as

\[
\begin{aligned}
\int_0^T \int_{\Omega} \left\{ (1 - 2ku) \ddot{u} \phi + c^2 \nabla u \cdot \nabla \phi + b(1 - \delta) \nabla \dot{u} \cdot \nabla \phi \\
+ b \delta |\nabla \dot{u}|^{q-1} \nabla \dot{u} \cdot \nabla \phi - 2k \dot{u}^2 \phi \right\} \, dx \, ds = 0
\end{aligned}
\]

(4.3)

holds for all test functions \(\phi \in \tilde{X} = L^2(0, T; W_0^{1,q+1}(\Omega))\), with \((u, \dot{u}) = (u_0, u_1)\). The following well-posedness result holds:
Chapter 4. Local well-posedness results

Proposition 4.1. \cite{9}, Theorem 2.3 Let assumptions (4.2) hold. For any $T > 0$ there is a \(\kappa_T > 0\) such that for all \(u_0, u_1 \in W^{1,q+1}_0(\Omega)\) with

\[
|u_1|^2_{L^2(\Omega)} + |\nabla u_0|^2_{L^2(\Omega)} + |\nabla u_1|^2_{L^2(\Omega)} + |\nabla u_1|_{L^{q+1}(\Omega)}^{q+1} + |\nabla u_0|_{L^{q+1}(\Omega)}^{q+1} \leq \kappa_T^2,
\]

there exists a weak solution \(u \in W \subset X = H^2(0, T; L^2(\Omega)) \cap W^{1,\infty}(0, T; W^{1,q+1}_0(\Omega))\) of (4.1), and

\[
W = \{v \in X : \|\tilde{v}\|_{L^2(0,T;L^2(\Omega))} \leq \bar{m} \wedge \|\tilde{v}\|_{L^\infty(0,T;L^2(\Omega))} \leq \bar{m}
\wedge \|\tilde{v}\|_{L^{q+1}(0,T;L^{q+1}(\Omega))} \leq M \wedge (v, \tilde{v}) = (u_0, u_1)\},
\]

with

\[
2|k|C_0^\Omega W^{1,q+1}_{0,q+1,\infty}(\kappa_T + T^{\frac{q}{q+1}}M) < 1,
\]

and \(\bar{m}\) sufficiently small, and \(u\) is unique in \(W\).

In \cite{9}, the issue of possible degeneracy of the Westervelt equation due to the factor \(1 - 2ku\) is resolved by the means of the embedding \(W^{1,q+1}_0(\Omega) \hookrightarrow L^\infty(\Omega)\), valid for \(q > d - 1\), and the following estimate

\[
|u(x,t)| \leq C_0^\Omega W^{1,q+1}_{0,q+1,\infty}|\nabla u(t)|_{L^{q+1}(\Omega)}
\leq C_0^\Omega W^{1,q+1}_{0,q+1,\infty} |\nabla u_0|_{L^{q+1}(\Omega)} + \int_0^t \nabla u \, ds |_{L^{q+1}(\Omega)}
\leq C_0^\Omega W^{1,q+1}_{0,q+1,\infty} \left(|\nabla u_0|_{L^{q+1}(\Omega)} + \left(t^q \int_0^t \int_\Omega |\nabla u(y, s)|_{L^{q+1}(\Omega)}^{q+1} \, dy \, ds \right)^{\frac{1}{q+1}} \right),
\]

which leads to the bound

\[
1 - a_0 < 1 - 2ku < 1 + a_0,
\]

\[
a_0 := 2|k|C_0^\Omega W^{1,q+1}_{0,q+1,\infty} |\nabla u_0|_{L^{q+1}(\Omega)} + T^{\frac{q}{q+1}} |\nabla u|_{L^{q+1}(0,T;L^{q+1}(\Omega))}.
\]

Due to the embedding \(W^{1,q+1}(\Omega) \hookrightarrow C^{0,1-\frac{d}{q+1}}(\Omega)\), we also know that \(u\) is Hölder continuous in space, i.e. \(u \in C^{0,1}(0,T;C^{0,1-\frac{d}{q+1}}(\Omega))\).

4.2 Neumann as well as absorbing boundary conditions

In this section \(\partial \Omega\) is assumed to be a disjoint union of \(\Gamma\) and \(\tilde{\Gamma}\). We denote by \(n\) the outward unit normal vector. In our case, the design of the nonlinear absorbing and inhomogeneous Neumann boundary conditions is influenced by the presence of the nonlinear strong damping in the equation (2.4). We will study initial boundary value problems of the following type:

\[
\begin{cases}
(1 - 2ku)\ddot{u} - c^2\Delta u - b \text{div}(\delta|\nabla \dot{u}|^{q-1})\nabla \dot{u} + \beta \dot{u} = 2k(\ddot{u})^2 \text{ in } \Omega \times (0,T), \\
c^2\frac{\partial u}{\partial n} + b((1 - \delta) + \delta|\nabla \dot{u}|^{q-1})\frac{\partial u}{\partial n} = g \text{ on } \Gamma \times (0,T), \\
\alpha \dot{u} + c^2\frac{\partial u}{\partial n} + b((1 - \delta) + \delta|\nabla \dot{u}|^{q-1})\frac{\partial u}{\partial n} = 0 \text{ on } \tilde{\Gamma} \times (0,T), \\
(u, \dot{u}) = (u_0, u_1) \text{ on } \bar{\Omega} \times \{t=0\},
\end{cases}
\]

(4.5)
4.2. Neumann as well as absorbing boundary conditions

\begin{align}
\begin{cases}
(1 - 2k\omega)\ddot{u} - c^2\Delta u - b\text{div}(\delta\nabla \dot{u}|^{q-1})\nabla \dot{u}) + \gamma|\dot{u}|^{q-1}\dot{u} \\
= 2k(\dot{u})^2 \text{ in } \Omega \times (0, T], \\
c^2\frac{\partial u}{\partial n} + b((1 - \delta) + \delta|\nabla \dot{u}|^{q-1})\frac{\partial u}{\partial n} = g \text{ on } \Gamma \times (0, T], \\
\alpha\dot{u} + c^2\frac{\partial u}{\partial n} + b((1 - \delta) + \delta|\nabla \dot{u}|^{q-1})\frac{\partial u}{\partial n} = 0 \text{ on } \hat{\Gamma} \times (0, T], \\
(u, \ddot{u}) = (u_0, u_1) \text{ on } \overline{\Omega} \times \{t = 0\}.
\end{cases}
\end{align}

(4.6)

Compared to the Westervelt equation with strong nonlinear damping (2.4) we have discussed so far, these equations have additional lower order damping terms. We assume that parameters \( \beta \) and \( \gamma \) are nonnegative; the case \( \beta = \gamma = 0 \) reduces them back to (2.4). We will in this way also look into the possible introduction of these lower order linear and nonlinear damping terms to equation (2.4), this becomes beneficial when deriving energy estimates.

The additional difficulty as compared to the Dirichlet case lies in not being able to use Poincare’s inequality for functions in \( W^{1,q+1}_0(\Omega) \), therefore we always have to find appropriate estimates of the zero order space derivative terms of \( u \) to be combined with the first order ones and employed in the trace estimates of the \( g \) terms arising from multiplication with \( \dot{u} \) and \( \ddot{u} \). As a matter of fact, we will try to tune these combinations to minimize the restrictions on \( T \) and on the norms of the data. This will lead to different combinations in the cases \( \beta = \gamma = 0 \) (Propositions 4.3, 4.6, Theorem 4.10), \( \beta > 0 \) (Proposition 4.7, Theorem 4.12), \( \gamma > 0 \) (Proposition 4.8, Theorem 4.13).

Our results will hold for \( \alpha \) assumed to be nonnegative, and they remain valid also in case \( \hat{\Gamma} = \emptyset \), that is to say, with Neumann boundary conditions on the whole boundary. Note that in the case of \( b = 0 \) and \( \alpha = c \) the absorbing conditions prescribed in (4.5) and (4.6) would reduce to the standard linear absorbing boundary conditions of the form \( \ddot{u} + c\frac{\partial u}{\partial n} = 0 \).

In what is to follow, we will denote by \( C_1^{tr} \) the norm of the trace mapping

\[ \text{tr}_\Gamma : W^{1,q+1}(\Omega) \rightarrow W^{-1,\frac{q+1}{q+1}}(\Gamma), \]

and by \( C_2^{tr} \) the norm of the trace mapping \( \text{tr}_\Gamma : H^1(\Omega) \rightarrow H^{-1/2}(\Gamma) \) (with \( C_1^{tr} = C_2^{tr} \) for \( q = 1 \)).

We will begin by looking at problems (4.5) and (4.6) with \( \beta = \gamma = 0 \) (in other words, an initial-boundary value problem for the equation (2.4)):

\begin{align}
\begin{cases}
(1 - 2k\omega)\ddot{u} - c^2\Delta u - b\text{div}(\delta\nabla \dot{u}|^{q-1})\nabla \dot{u}) = 2k(\dot{u})^2 \text{ in } \Omega \times (0, T], \\
c^2\frac{\partial u}{\partial n} + b((1 - \delta) + \delta|\nabla \dot{u}|^{q-1})\frac{\partial u}{\partial n} = g \text{ on } \Gamma \times (0, T], \\
\alpha\dot{u} + c^2\frac{\partial u}{\partial n} + b((1 - \delta) + \delta|\nabla \dot{u}|^{q-1})\frac{\partial u}{\partial n} = 0 \text{ on } \hat{\Gamma} \times (0, T], \\
(u, \ddot{u}) = (u_0, u_1) \text{ on } \overline{\Omega} \times \{t = 0\},
\end{cases}
\end{align}

(4.7)

and then later on consider the addition of lower order damping terms.

4.2.1 Partially linearized model. Following the approach in [9], we will first consider a partially linearized version of the PDE in (4.7), where nonlinearity remains only in the damping
Definition 4.2. (Weak solution) Let $a \in L^\infty(0,T;L^\infty(\Omega))$, $\dot{a} \in L^\infty(0,T;L^2(\Omega))$, $f \in L^\infty(0,T;L^4(\Omega))$, $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $g \in L^{\frac{q+1}{q}}(0,T;W^{-\frac{q}{q+1},\frac{q+1}{q}}(\Gamma))$. We say that $u \in \{ v : v \in H^1(0,T;H^1(\Omega)) \land \dot{v} \in L^{q+1}(0,T;W^{1,q+1}(\Omega)) \}$ is a weak solution of (4.8) if the following identity holds

$$
-\int_0^T \int_{\Omega} u(\dot{a}w + a\dot{w}) \, dx \, ds + \int_0^T \int_{\Omega} \left\{ a^2 \nabla u \cdot \nabla w + b((1 - \delta) + \delta |\nabla \dot{u}|^{q-1}) \nabla \dot{u} \cdot \nabla w \right\} \, dx \, ds \\
+ \alpha \int_0^T \int_{\Gamma} \dot{w} \, dx \, ds
= -\int_0^T \int_{\Omega} f \dot{w} \, dx \, ds + \int_0^T \int_{\Gamma} g w \, dx \, ds - \int_{\Omega} a(0) u_1 w(0) \, dx,
$$

and $u(0) = u_0$, for all $w \in H^1(0,T;W^{1,q+1}(\Omega))$, $w(T) = 0$.

Proposition 4.3. Let $T > 0$, $c^2$, $b > 0$, $\alpha \geq 0$, $\delta \in (0,1)$, $q \geq 1$ and assume that

(i) $a \in L^\infty(0,T;L^\infty(\Omega))$, $\dot{a} \in L^\infty(0,T;L^2(\Omega))$, $0 < a \leq a(x,t) \leq \bar{a}$,

(ii) $f \in L^\infty(0,T;L^4(\Omega))$,

(iii) $g \in L^{\frac{q+1}{q}}(0,T;W^{-\frac{q}{q+1},\frac{q+1}{q}}(\Gamma))$,

(iv) $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$,

with

$$
\|f - \frac{1}{2} \dot{a}\|_{L^\infty(0,T;L^2(\Omega))} \leq \dot{b} < \min \left\{ \frac{b(1 - \delta)}{2(C^1\Omega L^4)^2}, \frac{\bar{a}}{4T(C^2\Omega L^4)^2} \right\}.
$$

Then (4.8) has a unique weak solution in the sense of Definition 4.2.

$$
u \in \tilde{X} := \{ v : v \in H^1(0,T;H^1(\Omega)) \land \dot{v} \in L^{q+1}(0,T;W^{1,q+1}(\Omega)) \}
$$

and $u$ satisfies the energy estimate

$$
\begin{align*}
&\left[ \frac{\alpha}{4} - \frac{\dot{b}(C^1\Omega L^4)^2}{2T} - \epsilon_0 \right] \|\dot{u}\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{c^2}{4} \|\nabla u\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{\alpha}{2} \|\dot{u}\|^2_{L^2(0,T;L^2(\Gamma))} \\
&+ \left[ \frac{b(1 - \delta)}{2} - \frac{\dot{b}(C^1\Omega L^4)^2}{2T} \right] \|\nabla \dot{u}\|_{L^2(0,T;L^2(\Omega))}^2 + \left[ \frac{b\delta}{2} - \epsilon_1 \right] \|\nabla \dot{u}\|^2_{L^{q+1}(0,T;L^{q+1}(\Omega))} \\
&\leq \frac{\pi}{2} |u_1|^2_{L^2(\Omega)} + \frac{c^2}{2} \|\nabla u_0\|^2_{L^2(\Omega)} + \frac{1}{4\epsilon_0} (C^1\Omega C^2)^2 \|g\|^2_{L^1(0,T;W^{-\frac{q}{q+1},\frac{q+1}{q}}(\Gamma))} \\
&+ C(\epsilon_1, q + 1)(C^1\Omega (1 + C_P)) \frac{q+1}{T} \|g\|^2_{L^{q+1}(0,T;W^{-\frac{q}{q+1},\frac{q+1}{q}}(\Gamma))}.
\end{align*}
$$

Chapter 4. Local well-posedness results
4.2. Neumann as well as absorbing boundary conditions

for some constants

\begin{equation}
0 < \epsilon_0 < \frac{a}{4} - \hat{b}(C^q_{H^1,L^4})^2 T, \quad 0 < \epsilon_1 < \frac{b\delta}{2},
\end{equation}

and depends Hölder continuously (in $\tilde{X}$) on the initial and boundary data. Moreover, if $a$ is time independent, then $\ddot{u} \in L^\frac{q+1}{q}(0,T; (W^{1,q+1}(\Omega))^*)$.

Remark 4.4. In case $\hat{\Gamma} = \emptyset$, estimate (4.11) and all forthcoming estimates hold with $\alpha$ set to zero.

Proof. Consider the problem of finding $u$ such that

\begin{equation}
\begin{aligned}
&\int_{\Omega} \left\{ a(t) \ddot{u}(t) w + c^2 \nabla u(t) \cdot \nabla w + b((1 - \delta) + \delta|\nabla \dot{u}(t)|^{q-1}) \nabla \dot{u}(t) \cdot \nabla w \right\} dx \\
&\quad + \alpha \int_{\Gamma} \dot{u}(t) w dx \\
&\quad = - \int_{\Omega} f(t) \dot{u}(t) w dx + \int_{\Gamma} g(t) w dx, \quad \forall w \in W^{1,q+1}(\Omega), \text{ a.e. } t \in [0,T],
\end{aligned}
\end{equation}

with initial conditions $(u_0, u_1)$.

We will use the standard Faedo-Galerkin method (see for instance [20, Section 7.2] for the case of second-order linear hyperbolic equations and [9, Section 2] for the problem (4.8) with homogeneous Dirichlet boundary conditions), where we will first construct approximations of the solution, and then by obtaining energy estimates guarantee weak convergence of these approximations.

**Step 1: Galerkin approximations.** We start by proving existence and uniqueness of a solution for a finite-dimensional approximation of (4.13). We choose smooth functions $w_m = w_m(x)$, $m \in \mathbb{N}$ such that

- $\{w_m\}_{m \in \mathbb{N}}$ is an orthonormal basis of $L^2_{\tilde{a}}(\Omega)$,
- $\{w_m\}_{m \in \mathbb{N}}$ is a basis of $W^{1,q+1}(\Omega)$,
- $\{w_m|_{\Gamma}\}_{m \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Gamma)$,

where $L^2_{\tilde{a}}$ is the weighted $L^2$-space based on the inner product

$$\langle f, g \rangle_{L^2_{\tilde{a}}(\Omega)} := \int_{\Omega} \tilde{a} fg dx,$$

with $\tilde{a} = \frac{1}{T} \int_0^T a(t) dt$. Next, we construct a sequence of finite dimensional subspaces $V_n$ of $L^2_{\tilde{a}}(\Omega) \cap W^{1,q+1}(\Omega)$,

$$V_n = \text{span}\{w_1, w_2, \ldots, w_n\}.$$ 

Clearly, $V_n \subseteq V_{n+1}$, $V_n \subseteq L^2_{\tilde{a}}(\Omega) \cap W^{1,q+1}(\Omega)$ and $\bigcup_{n \in \mathbb{N}} V_n = W^{1,q+1}(\Omega)$. Let

$$u_n(t) = \sum_{j=1}^{n} r_j(t) w_j.$$
Let \( \{u_{0,n}\}_{n \in \mathbb{N}}, \{u_{1,n}\}_{n \in \mathbb{N}} \) be sequences such that
\[
\begin{align*}
u_{0,n} &\in V_n, \quad u_{0,n} \to u_0 \text{ in } H^1(\Omega), \\
u_{1,n} &\in V_n, \quad u_{1,n} \to u_1 \text{ in } L^2(\Omega).
\end{align*}
\]
(4.14)

We can now consider a sequence of discretized versions of (4.13):
\[
\int_{\Omega} \left\{ a(t) \dot{u}_n(t) w_j + c^2 \nabla u_n(t) \cdot \nabla w_j + b((1 - \delta) \Delta u_n(t) + \delta \nabla u_n(t)) |\nabla u_n(t)|^{q-1} \right\} \nabla \dot{u}_n(t) \cdot \nabla w_j \right\} \, dx
\]
(4.15)
\[
\int_{\Omega} a(t) \dot{u}_n(t) w_j + \alpha \int_{\Gamma} \dot{u}_n(t) w_j \, dx
\]
\[
= - \int_{\Omega} f(t) \dot{u}_n(t) w_j \, dx + \int_{\Gamma} g(t) w_j \, dx, \quad \text{for all } j = 1, \ldots, n, \text{ a.e. } t \in [0, T],
\]
with \( u_n(0) = u_{0,n}, \dot{u}_n(0) = u_{1,n} \).

For each \( n \in \mathbb{N} \), we face an initial-value problem for a second order system for \( (r_j)_{j=1,\ldots,n} \) of \( n \) nonlinear ordinary differential equations. After reformulating it as a first order system, according to the existence result for solutions of ordinary differential equations given in Theorem 3.14, we can conclude that there exists a solution of (4.15) on some sufficiently short time interval \([0, T_n], T_n \leq T\). The energy estimate we will derive next (with the right hand side independent of \( n \)) will allow us to extend the solution to the whole \([0, T]\).

**Step 2: Energy estimate.** Testing (4.15) with \( w_n = \dot{u}_n(t) \in V_n \) and integrating with respect to time results in
\[
\frac{1}{2} \left[ \int_{\Omega} a(\dot{u}_n)^2 \, dx + c^2 |\nabla u_n|_{L^2(\Omega)}^2 \right]_0^t + \alpha \int_0^t \int_{\Gamma} |\dot{u}_n|^2 \, dx \, ds
\]
\[
+ b \int_0^t \int_{\Omega} (1 - \delta) |\nabla u_n|^{q-1} |\nabla \dot{u}_n|^2 \, dx \, ds
\]
\[
= - \int_0^t \int_{\Omega} (1 - \frac{1}{2} \dot{u}_n)^2 \, dx \, ds + \int_0^t \int_{\Gamma} g \dot{u}_n \, dx \, ds
\]
\[
\leq \|f - \frac{1}{2} \dot{u}_n\|_{L^\infty(0,T;L^2(\Omega))} \int_0^t |\dot{u}_n|^2_{L^q(\Omega)} \, ds + \int_0^t \int_{\Gamma} g \dot{u}_n \, dx \, ds.
\]
(4.16)

For estimating the boundary integral appearing on the right side, we will make use of duality between \( W^{-\frac{1}{q+1}}(\Gamma) \) and \( W_{-\frac{1}{q+1}}\frac{q+1}{q} (\Gamma) \), and estimate (3.8) to obtain
\[
\int_0^t \int_{\Gamma} g \dot{u}_n \, dx \, ds
\]
\[
\leq \int_0^t |\dot{u}_n(s)|_{W^{-\frac{1}{q+1}}(\Gamma)} \|g(s)\|_{W^{\frac{q+1}{q}}(\Gamma)} \, ds
\]
\[
\leq C_{1r}^{|r|} \left| \int_0^t |\dot{u}_n(s)|_{W^{\frac{q+1}{q+1}}(\Omega)} \|g(s)\|_{W^{\frac{q}{q+1}}(\Gamma)} \, ds \right|
\]
\[
\leq C_{1r}^{|r|} \left[ (1 + C_P) \|\nabla \dot{u}_n(s)\|_{L^q(\Omega)} + C_P^{|r|} \|\dot{u}_n(s)\|_{L^2(\Omega)} \right] \|g(s)\|_{W^{\frac{q}{q+1}}(\Gamma)} \, ds
\]
\[
\leq \epsilon_1 \|\nabla \dot{u}_n\|_{L^q(0,T;L^q(\Omega))} + \epsilon_0 \|\dot{u}_n\|_{L^q(0,T;L^q(\Omega))}^2
\]
\[
+ C(\epsilon_1, q + 1)(C_{1r}^{|r|} (1 + C_P))^{\frac{q+1}{q}} \|g\|_{L^{q+1}(0,T;W^{-\frac{1}{q+1}}(\Gamma))}^{\frac{q+1}{q}}
\]
(4.17)
with $\epsilon_0, \epsilon_1 > 0$. By taking the essential supremum with respect to $t$ in (4.16) and employing the embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ (see (3.2)), as well as the inequality

$$
\|\dot{u}\|_{L^2(0,T;L^2(\Omega))} \leq T \|\ddot{u}\|_{L^\infty(0,T;L^2(\Omega))},
$$

we obtain the estimate

$$
\left[ \frac{a}{4} - \hat{b}(\mathcal{Q}_{H^1;L^4})^2 T - \epsilon_0 \right] \|\ddot{u}_n\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{\epsilon_1^2}{4} \|\nabla u_n\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{\epsilon_1^2 b}{2} \|\nabla u_n\|^2_{L^2(0,T;L^2(\Gamma))} + \frac{\epsilon_1}{2} \|\nabla u_0\|^2_{L^2(\Omega)}
$$

$$
\leq C(\epsilon_1, q + 1)(\mathcal{Q}_{H^1;L^4}) q + 1 \frac{a}{4} \|\dot{u}\|_{L^4(0,T;W^{-\frac{q}{2} + \frac{q+1}{4}}(\Omega))} + \frac{\epsilon_1^2}{2} \|\nabla u_0\|^2_{L^2(\Omega)}.
$$

We choose $\epsilon_0, \epsilon_1$ small enough so that coefficients appearing in the estimate remain positive. Since we assumed that $g \in L^{\frac{q+1}{2}}(0, T; W^{-\frac{q}{2} + \frac{q+1}{4}}(\Gamma))$, we can conclude that the sequence of Galerkin approximations $(u_n)_{n \in \mathbb{N}}$ is bounded in the space

$$
\tilde{X} := \{ v : v \in H^1(0, T; H^1(\Omega)) \cap W^{1,\infty}(0, T; L^2(\Omega)) \wedge \dot{v} \in L^{q+1}(0, T; W^{1,q+1}(\Omega)) \}.
$$

**Estimate for $a\ddot{u}_n$.** Let $v \in W^{1,q+1}(\Omega)$. We can decompose it as follows:

$$
v = u_n + z_n, \quad v_n \in V_n, \quad z_n \in V_n^{1,L^2}. $$

It can be shown that $|v_n|_{W^{1,q+1}(\Omega)} \leq |v|_{W^{1,q+1}(\Omega)}$ (see [3] Section 2). By utilizing orthogonality (and under the assumption that $a$ is time independent), we then have

$$
\int_\Omega a\ddot{u}_n v \, dx = \int_\Omega a\ddot{u}_n v_n \, dx
$$

\[- \int_\Omega \{ c^2 \nabla u_n \cdot \nabla v + b((1 - \delta) + \delta) \nabla \dot{u}_n \nabla \ddot{u}_n \cdot \nabla v \} \, dx
$$

\[- \int_\Omega \{ c^2 \nabla u_n \cdot \nabla v + b(1 - \delta) \nabla \dot{u}_n \nabla \ddot{u}_n \cdot \nabla v \} \, dx
$$

\[- \int_\Omega \{ c^2 \nabla u_n \cdot \nabla v + b(1 - \delta) \nabla \dot{u}_n \nabla \ddot{u}_n \cdot \nabla v \} \, dx
$$

\[- \int_\Omega \{ c^2 \nabla u_n \cdot \nabla v + b(1 - \delta) \nabla \dot{u}_n \nabla \ddot{u}_n \cdot \nabla v \} \, dx
$$

\[- \int_\Omega \{ c^2 \nabla u_n \cdot \nabla v + b(1 - \delta) \nabla \dot{u}_n \nabla \ddot{u}_n \cdot \nabla v \} \, dx
$$

\[- \int_\Omega \{ c^2 \nabla u_n \cdot \nabla v + b(1 - \delta) \nabla \dot{u}_n \nabla \ddot{u}_n \cdot \nabla v \} \, dx
$$

for a.e. $t \in [0, T]$. From here we obtain

$$
|a\ddot{u}_n|_{W^{1,q+1}(\Omega)}, \leq C \left\{ |\nabla u_n|_{L^2(\Omega)} + |\nabla \dot{u}_n|_{L^2(\Omega)} + |\nabla \ddot{u}_n|_{L^2(\Omega)} \right\}.
$$
for some $C > 0$ independent of $n$. By raising the inequality to the power of $\frac{q+1}{q}$ and integrating over $(0, T)$ we can achieve that $a\dot{u}_n \in L^{\frac{q+1}{q}}(0, T; (W^{1,q+1}(\Omega))^*)$, and (thanks to (4.19)) with a uniform bound with respect to $n$.

It follows from (4.19) that

\begin{align}
\text{(4.20)} & \quad (\dot{u}_n)_{n \in \mathbb{N}} \text{ is uniformly bounded in } L^2(0, T; L^2(\Omega)), \\
\text{(4.21)} & \quad (\nabla \dot{u}_n)_{n \in \mathbb{N}} \text{ is uniformly bounded in } L^{q+1}(0, T; L^{q+1}(\Omega)), \\
\text{(4.22)} & \quad (|\nabla \dot{u}_n|^{q-1} \nabla \dot{u}_n)_{n \in \mathbb{N}} \text{ is uniformly bounded in } L^{\frac{q+1}{q}}(0, T; L^{\frac{q+1}{q}}(\Omega)), \text{ and} \\
\text{(4.23)} & \quad (\dot{u}_n|_{\Gamma})_{n \in \mathbb{N}} \text{ is uniformly bounded in } L^2(0, T; L^2(\Gamma)),
\end{align}

which are all reflexive Banach spaces.

**Step 3: Convergence of Galerkin approximations.** Due to (4.20)-(4.23) there exists a weakly convergent subsequence of $\{u_n\}_{n \in \mathbb{N}}$, which we still denote $\{u_n\}_{n \in \mathbb{N}}$, and a $u$ such that

\begin{align}
\text{(4.24)} & \quad \dot{u}_n \rightharpoonup \dot{u} \text{ in } L^2(0, T; L^2(\Omega)), \\
\text{(4.25)} & \quad \nabla \dot{u}_n \rightharpoonup \nabla \dot{u} \text{ in } L^{q+1}(0, T; L^{q+1}(\Omega)), \\
\text{(4.26)} & \quad |\nabla \dot{u}_n|^{q-1} \nabla \dot{u}_n \rightharpoonup |\nabla \dot{u}|^{q-1} \nabla \dot{u} \text{ in } L^{\frac{q+1}{q}}(0, T; L^{\frac{q+1}{q}}(\Omega)), \\
\text{(4.27)} & \quad \dot{u}_n|_{\Gamma} \rightharpoonup \dot{u}|_{\Gamma} \text{ in } L^2(0, T; L^2(\Gamma)),
\end{align}

where in (4.26) we have made use of monotonicity (3.12) and Minty’s lemma. Due to the embedding $H^1(0, T) \hookrightarrow C(0, T)$, we know that $u_n(t) \rightharpoonup u(t)$ in $L^2(\Omega)$, for all $t \in [0, T]$. Therefore $u(0) = u_0$.

Our task next is to prove that the weak limit $u$ solves (4.13) in the sense of Definition 4.2. Fix $m \in \mathbb{N}$ and let $\phi_m \in C^\infty(0, T; V_m) \subset H^1(0, T; W^{1,q+1}(\Omega))$ such that $\phi_m(T) = 0$. For any $n \geq m$, by $V_m \subseteq V_n$ we have

\begin{align}
\int_0^T \int_\Omega \left\{ a\ddot{u}\phi_m + c^2 \nabla u \cdot \nabla \phi_m + b\left( (1 - \delta) + \delta|\nabla \dot{u}|^{q-1} \right) \nabla \dot{u} \cdot \nabla \phi_m \\
+ f\dot{u} \phi_m \right\} dx \, ds + \alpha \int_0^T \int_\Gamma \dot{u} \phi_m dx \, ds - \int_0^T \int_\Gamma g \phi_m dx \, ds \\
= -\int_0^T \int_\Omega \left[ \dot{u} - \dot{u}_n \right] \frac{d}{dt} \left( a\phi_m \right) dx \, ds - \int_0^T \int_\Omega \left[ u_1 - u_{1,n} \right] a(0) \phi_m(0) dx \, ds \\
+ c^2 \int_0^T \int_\Omega |\nabla u - \nabla u_n| \cdot \nabla \phi_m dx \, ds + b(1 - \delta) \int_0^T \int_\Omega |\nabla \dot{u} - \nabla \dot{u}_n| \cdot \nabla \phi_m dx \, ds \\
+ b\delta \int_0^T \int_\Omega |\nabla \dot{u}|^{q-1} \nabla \dot{u} - |\nabla \dot{u}_n|^{q-1} \nabla \dot{u}_n| \cdot \nabla \phi_m dx \, ds \\
+ \int_0^T \int_\Gamma \left[ \dot{u} - \dot{u}_n \right] f \phi_m dx \, ds + \alpha \int_0^T \int_\Gamma \left[ \dot{u} - \dot{u}_n \right] \phi_m dx \, ds \rightarrow 0 \text{ as } n \rightarrow \infty,
\end{align}

due to (4.24)-(4.27) and (4.14). Since $\bigcup_{m \in \mathbb{N}} V_m$ is dense in $W^{1,q+1}(\Omega)$, $u$ indeed is a weak solution of (4.8).
4.2. Neumann as well as absorbing boundary conditions

By testing problem (4.13) with \( \hat{u} \), integrating with respect to time, and proceeding as in Step 2, we can conclude that this weak limit satisfies the estimate (4.19) with \( u_n \) replaced by \( u \).

Step 4: Uniqueness. To confirm uniqueness, note that the difference \( \hat{u} = u^1 - u^2 \) between any two weak solutions \( u^1, u^2 \) of (4.8) is a weak solution of the problem

\[
\begin{align*}
& a\ddot{u} - c^2\Delta\ddot{u} - b(1 - \delta)\Delta\dot{u} - b\delta\text{div}(|\nabla u|^q - 1\nabla u^1 - |\nabla u^2|^q - 1\nabla u^2) + f\dot{u} = 0 \quad \text{in } \Omega, \\
& c^2\frac{\partial\ddot{u}}{\partial n} + b(1 - \delta)\frac{\partial\dot{u}}{\partial n} + b\delta(|\nabla u|^q - 1\frac{\partial u^1}{\partial n} - |\nabla u^2|^q - 1\frac{\partial u^2}{\partial n}) = 0 \quad \text{on } \Gamma, \\
& \alpha\dot{u} + c^2\frac{\partial u}{\partial n} + b(1 - \delta)\frac{\partial u}{\partial n} + b\delta(|\nabla u|^q - 1\frac{\partial u^1}{\partial n} - |\nabla u^2|^q - 1\frac{\partial u^2}{\partial n}) = 0 \quad \text{on } \hat{\Gamma}, \\
& (\hat{u}, \dot{u})|_{t=0} = (0, 0).
\end{align*}
\]

Multiplication of (4.28) by \( \hat{u} \) yields

\[
\frac{1}{2} \int_\Omega a(\dot{u})^2 \, dx + c^2|\nabla \dot{u}|^2_{L^2(\Omega)} \right|^t_0 + b(1 - \delta) \int_\Omega \nabla \dot{u}^2 \, dx \, ds + \alpha \int_0^t |\dot{u}|^2_{L^2(\Gamma)} \, ds \\
+ \int_0^t \int_\Omega (f - \frac{1}{2}\dot{u})(\hat{u})^2 \, dx \, ds \leq 0,
\]

since due to the inequality (3.12) we have

\[
b\delta \int_0^t \int_\Omega (|\nabla u|^q - 1|\nabla u^1 - |\nabla u^2|^q - 1\nabla u^2)| \cdot \nabla \dot{u} \, dx \, ds \geq 0.
\]

From here, we conclude that \( \hat{u} = 0 \) and \( \nabla \hat{u} = 0 \) almost everywhere, which results in the solution being unique up to an additive constant. The initial condition \( \hat{u}|_{t=0} = 0 \) provides us with uniqueness.

Step 5: Continuous dependence on the initial and boundary data. If \( u^1 \) and \( u^2 \) are solutions corresponding to the problems with data \((u_0^1, u_1^1, g^1)\) and \((u_0^2, u_1^2, g^2)\), respectively, then the difference \( \hat{u} = u^1 - u^2 \) weakly satisfies the following problem

\[
\begin{align*}
& a\ddot{u} - c^2\Delta\ddot{u} - b(1 - \delta)\Delta\dot{u} - b\delta\text{div}(|\nabla u|^q - 1\nabla u^1 - |\nabla u^2|^q - 1\nabla u^2) + f\dot{u} = 0 \quad \text{in } \Omega, \\
& c^2\frac{\partial\ddot{u}}{\partial n} + b(1 - \delta)\frac{\partial\dot{u}}{\partial n} + b\delta(|\nabla u|^q - 1\frac{\partial u^1}{\partial n} - |\nabla u^2|^q - 1\frac{\partial u^2}{\partial n}) = g^1 - g^2 \quad \text{on } \Gamma, \\
& \alpha\dot{u} + c^2\frac{\partial u}{\partial n} + b(1 - \delta)\frac{\partial u}{\partial n} + b\delta(|\nabla u|^q - 1\frac{\partial u^1}{\partial n} - |\nabla u^2|^q - 1\frac{\partial u^2}{\partial n}) = 0 \quad \text{on } \hat{\Gamma}, \\
& (\hat{u}, \dot{u})|_{t=0} = (u_0^1 - u_0^2, u_1^1 - u_1^2).
\end{align*}
\]

Multiplication by \( \hat{u} \) and integration with respect to space and time produces

\[
\frac{1}{2} \int_\Omega a(\dot{u})^2 \, dx + c^2|\nabla \dot{u}|^2_{L^2(\Omega)} \right|^t_0 + b(1 - \delta) \int_\Omega \nabla \dot{u}^2 \, dx \, ds + \alpha \int_0^t |\dot{u}|^2_{L^2(\Gamma)} \, ds \\
\leq - \int_0^t \int_\Omega (f - \frac{1}{2}\dot{u})(\hat{u})^2 \, dx \, ds + \int_0^t \int_\Omega (g^1 - g^2)\dot{u} \, dx \, ds.
\]
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Then, analogously as before, we can show that the following estimate holds for sufficiently large $C > 0$:

\[
\begin{align*}
\|\hat{u}\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\nabla \hat{u}\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\nabla \hat{u}\|_{L^2(0,T;L^2(\hat{\Gamma}))}^2 + \|\hat{u}\|_{L^2(0,T;L^2(\hat{\Gamma}))}^2 &
\leq C(\|g^1 - g^2\|_{L^\infty(0,T;W^{-\frac{\alpha}{2},\alpha\frac{1}{4}}(\Gamma))}^2 + \|g^1 - g^2\|_{L^1(0,T;W^{-\frac{\alpha}{2},\alpha\frac{1}{4}}(\Gamma))}^2 + |u_1|^2_{L^2(\Omega)}^2 + |u_0|^2_{L^2(\Omega)}^2)
\end{align*}
\]

(4.31)

\[
\text{Remark 4.5. Note that due to the desire to include the case of having only inhomogeneous Neumann boundary conditions on the whole boundary in the model (4.8), that is to allow the case of $\hat{\Gamma}$ being of measure zero, inequality (4.18) was employed as a way of dominating the term $\|u\|_{L^2(0,T;L^2(\hat{\Gamma}))}$ and therefore dependence on final time $T > 0$ is present in the energy estimate (4.11). If we would to assume that $\hat{\Gamma}$ is of strictly positive measure, it would be possible to avoid this dependence on final time by employing the following analogue of Poincaré’s inequality:}

\[
|v|_{L^2(\Omega)} \leq C(\|\nabla v\|_{L^2(\Omega)} + |v|_{L^2(\hat{\Gamma})}), \quad v \in H^1(\Omega).
\]

\textbf{Proposition 4.6. Let $T > 0$, $c^2$, $b > 0$, $\alpha \geq 0$, $\delta \in (0,1)$, $q \geq 1$ and let assumptions (i) of Proposition 4.3 hold. If, in addition to (i), the following assumptions hold:}

(\textit{i})

- $\exists \hat{b} > 0 : \|f\|_{L^\infty(0,T;L^2(\Omega))} \leq \hat{b}$,
- $g \in L^\infty(0,T;W^{-\frac{\alpha}{2},\alpha\frac{1}{4}}(\Gamma))$, $\dot{g} \in L^\frac{\alpha+1}{\alpha}(0,T;W^{-\frac{\alpha}{2},\alpha\frac{1}{4}}(\Gamma))$,
- $u_1 \in W^{1,q+1}(\Omega),$

\textit{then problem (4.8) has a unique weak solution}

(4.32)

\[
u \in X := W^{1,\infty}(0,T;W^{1,q+1}(\Omega)) \cap H^2(0,T;L^2(\Omega)),
\]

and satisfies the energy estimate

\[
\begin{align*}
\mu \frac{a - \tau}{2} \|\hat{u}\|_{L^2(0,T;L^2(\Omega))}^2 + \mu \frac{b(1 - \delta)}{4} - \sigma \|\nabla \hat{u}\|_{L^\infty(0,T;L^2(\Omega))}^2 + \mu \frac{\alpha}{4} \|\hat{u}\|_{L^2(0,T;L^2(\hat{\Gamma}))}^2 &
\leq C(\|\nabla (g)\|_{L^{q+1}(0,T;L^{q+1}(\hat{\Gamma}))}^2 + |u_1|^2_{W^{1,q+1}(\Omega)} + |u_0|^2_{L^2(\hat{\Gamma})})
\end{align*}
\]

(4.33)

for some sufficiently small constants $\mu, \sigma, \tau, \eta > 0$, some large enough $\bar{C} > 0$, and

\[
C_T(g) := \sum_{s=0}^{1} \frac{d^s}{dt^s} \|g\|_{L^\infty(0,T;W^{-\frac{\alpha}{2},\alpha\frac{1}{4}}(\Gamma))}^2 + \sum_{s=0}^{1} \frac{d^s}{dt^s} \|g\|_{L^2(0,T;W^{-\frac{\alpha}{2},\alpha\frac{1}{4}}(\Gamma))}^2 + \|g\|_{L^\infty(0,T;W^{-\frac{\alpha}{2},\alpha\frac{1}{4}}(\Gamma))}^2.
\]

(4.34)
4.2. Neumann as well as absorbing boundary conditions

Proof. The proof is carried out as before, through Galerkin approximations in space. We will focus here on obtaining the higher order energy estimate. After discretizing in space as in the proof of Proposition 4.3, to obtain the higher order estimate (4.33), we will test (4.15) with \( w_n = \tilde{u}_n(t) \in V_n \) and then combine the result with the lower order estimate (4.11) we derived previously. Multiplication by \( \tilde{u}_n(t) \) and integration with respect to time produces

\[
\int_0^t \int_\Omega a(\tilde{u}_n)^2 dx \, ds + \left[ \frac{b(1 - \delta)}{2} |\nabla \tilde{u}_n|^2_{L^2(\Omega)} + \frac{b\delta}{q+1} |\nabla \tilde{u}_n|_{L^{q+1}(\Omega)}^{q+1} \right]_0^t + \frac{\alpha}{2} \int_\Gamma (\tilde{u}_n)^2 \, ds \quad (4.35)
\]

To estimate the boundary integral on the right side, we employ (3.8) to obtain

\[
\int_0^t \int_\Gamma g \tilde{u}_n \, ds \leq C (\eta, q + 1) \left( C^{(q+1)}(1 + C_P) \right)^{q+1} \|g\|_{L^\infty(0,T;W^{-\frac{q}{q+1}}(\Gamma))}^{\frac{q+1}{q}} + C(1, q + 1) \left( C^{(q+1)}(1 + C_P) \right)^{q+1} \|g\|_{L^\infty(0,T;W^{-\frac{q}{q+1}}(\Gamma))}^{\frac{q+1}{q}} + \epsilon_1 \|\nabla \tilde{u}_n\|_{L^{q+1}(0,T;L^{q+1}(\Omega))}^{q+1} \|g\|_{L^\infty(0,T;W^{\frac{q}{q+1}}(\Gamma))}^{q+1} \|g\|_{L^\infty(0,T;W^{q+1}(\Omega))}^{q+1} \| \nabla \tilde{u}_n \|_{L^{q+1}(0,T;L^{q+1}(\Omega))}^{q+1} \| \nabla \tilde{u}_n \|_{L^{q+1}(0,T;L^{q+1}(\Omega))}^{q+1}
\]

which together with

\[
\int_0^t \int_\Omega f \tilde{u}_n \tilde{u}_n \, ds \leq \frac{1}{2} \left( C^{(q+1)}_{H^1,L^q} \right)^2 \|f\|_{L^\infty(0,T;L^q(\Omega))} \left[ T\|\tilde{u}_n\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\nabla \tilde{u}_n\|_{L^2(0,T;L^2(\Omega))}^2 \right] + \frac{\alpha}{2} \|\tilde{u}_n\|_{L^2(0,T;L^2(\Omega))}^2
\]

and taking \( \sup \) in (4.35) leads to the estimate

\[
\frac{\alpha}{2} \|\tilde{u}_n\|_{L^2(0,T;L^2(\Omega))}^2 + \left( \frac{b(1 - \delta)}{4} - \sigma \right) \|\nabla \tilde{u}_n\|_{L^\infty(0,T;L^2(\Omega))}^2 + \frac{b\delta}{2(q+1)} \|\nabla \tilde{u}_n\|_{L^\infty(0,T;L^{q+1}(\Omega))}^{q+1} + \frac{\alpha}{4} \|\tilde{u}_n\|_{L^\infty(0,T;L^2(\Gamma))}^2 \leq c^2 \|\nabla \tilde{u}_n\|_{L^2(0,T;L^2(\Omega))}^2 + \sigma \|\nabla u_{n1}\|_{L^2(0,T;L^2(\Omega))}^2 + \frac{c^4}{4\sigma} \|\nabla \tilde{u}_n\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\nabla u_{0n}\|_{L^2(0,T;L^2(\Omega))}^2
\]
which are reflexive Banach spaces. From here, after proceeding as in Proposition 4.3, we can obtain

\[ (4.38) \]

\[ u(t) = u_0 + \int_0^t \Delta u(s) + b \nabla \nabla u(s) \, ds + \int_0^t f(s) \, ds \]

Addition of a linear lower order damping term. Let us now consider the boundary value problem (4.8) with an added lower order linear damping term:

\[ \alpha u + c^2 u_n + b((1 - \delta) + \delta |\nabla u|^q - 1) \nabla u + \beta u + f u = 0 \quad \text{on } \Gamma \times (0, T), \]

\[ (u, u_n) = (u_0, u_1) \quad \text{on } \Omega \times \{ t = 0 \}, \]

Since there are terms on the right side in (4.38) which cannot be dominated by the terms on the left hand side, we need to also employ the lower estimate (4.19). Adding (4.19) and \( \mu \) times (4.38) yields (4.33) with \( u \) replaced by \( u_n \), provided that we choose

\[ 0 < \tau < \varrho, \quad 0 < \eta < \frac{b\vartheta}{2(q + 1)}, \quad 0 < \sigma < \frac{b(1 - \delta)}{4}, \]

\[ 0 < \mu < \min \left\{ \frac{b(1 - \delta)}{2} - (C_{H^1,L^2}^\vartheta)^2 b_2 - \frac{a}{4} - (C_{H^2,L^2}^\vartheta)^2 c^2 + \epsilon_0, \frac{1}{2\tau} (C_{H^1,L^2}^\vartheta)^2 b_2 - \frac{a}{4} - (C_{H^2,L^2}^\vartheta)^2 c^2 + \epsilon_0, \frac{b\vartheta}{2q}, \epsilon_1 \right\}, \]

so that the coefficients in (4.33) are positive.

As by assumption \( g \in L^\infty(0, T; W^{-\frac{q+1}{q+1} \frac{q}{q+1}, \Gamma}) \) and \( \dot{g} \in L^\infty(0, T; W^{-\frac{q+1}{q+1} \frac{q}{q+1}, \Gamma}) \), \( (u_n)_{n \in \mathbb{N}} \) is a bounded sequence in

\[ X := W^1(0, T; W^{\vartheta \frac{q+1}{q+1}}(\Omega)) \cap H^2(0, T; L^2(\Omega)). \]

We further obtain

\[ (4.39) \]

\[ (u_n)_{n \in \mathbb{N}} \]

is uniformly bounded in \( L^2(0, T; L^2(\Omega)) \),

\[ (\nabla u_n)_{n \in \mathbb{N}} \]

is uniformly bounded in \( L^{q+1}(0, T; L^{q+1}(\Omega)) \),

\[ (\nabla^2 u_n)_{n \in \mathbb{N}} \]

is uniformly bounded in \( L^{q+1}(0, T; L^{q+1}(\Omega)) \) and

\[ (\nabla^3 u_n)_{n \in \mathbb{N}} \]

is uniformly bounded in \( L^2(0, T; L^2(\Omega)) \),

which are reflexive Banach spaces. From here, after proceeding as in Proposition 4.3, we can conclude that (4.13) has a unique solution \( u \in X \), which satisfies the estimate (4.33). \( \square \)

Addition of a linear lower order damping term. Let us now consider the boundary value problem (4.8) with an added lower order linear damping term:
where $\beta > 0$. This is a linearized version of the problem \([4.5]\) with nonlinearity appearing only through the damping term. The additionally introduced $\beta$—lower order term will allow us to remove restrictions on the final time $T$ in the estimates \([4.11]\) and \([4.33]\) (including the case of having only inhomogeneous Neumann boundary conditions). Indeed, by testing the equation with $\hat{u}$ (first in a discretized setting and then passing to the limit) and integrating with respect to space and time, we obtain

\[
\frac{1}{2} \int_0^T \int_\Omega a(\hat{u}^2) \, dx + c^2 |\nabla \hat{u}|^2_{L^2(\Omega)} \, dt + b \int_0^T \int_\Omega ((1 - \delta) + \delta |\nabla \hat{u}|^q-1) |\nabla \hat{u}|^2 \, dx \, ds \\
+ \beta \int_0^T \int_\Omega |\hat{u}|^2 \, dx \, ds + \alpha \int_0^T \int_\Gamma |\hat{u}|^2 \, ds \, ds \\
\leq \hat{b}(C^\Omega_{H^1,L^2})^2 \int_0^T |\hat{u}|^2_{H^1(\hat{\Gamma})} \, ds + \epsilon_0 \|\hat{u}\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{1}{4\epsilon_0}(C^\Omega_{H^1,L^2})^2 \|g\|^2_{L^1(0,T;W^{-\frac{q+1}{q}})} \\
+ \epsilon_1 \|\nabla \hat{u}\|_{L^2(0,T;L^{q+1}(\Omega))}^q + C(\epsilon_1, q + 1)(C^\Omega_{1 (1 + C_F)})^q \|g\|^q_{L^\frac{q+1}{q}(0,T;W^{-\frac{q+1}{q}})} \),
\]

which leads to the lower order energy estimate

\[
\left(\frac{\alpha}{4} - \epsilon_0 \right) \|\hat{u}\|^2_{L^\infty(0,T;L^2(\Omega))} + \left(\frac{b(1 - \delta)}{2} - \hat{b}(C^\Omega_{H^1,L^2})^2\right) \|\nabla \hat{u}\|^2_{L^2(0,T;L^2(\Omega))} \\
+ \frac{b\delta}{2} - \epsilon_1 \right) \|\nabla \hat{u}\|^2_{L^\infty(0,T;L^{q+1}(\Omega))} + \frac{\beta}{2} - \hat{b}(C^\Omega_{H^1,L^2})^2 \|\hat{u}\|^2_{L^2(0,T;L^2(\Omega))} \\
+ \frac{c^2}{4} \|\nabla \hat{u}\|^2_{L^\infty(0,T;L^2(\Omega))} + \alpha \|\hat{u}\|^2_{L^2(0,T;L^2(\Gamma))} \\
\leq \frac{\eta}{2} \|u_1\|^2_{L^2(\Omega)} + \frac{\gamma}{2} \|\nabla u_0\|^2_{L^2(\Omega)} + \frac{1}{4\epsilon_0}(C^\Omega_{1 (1 + C_F)})^q \|g\|^q_{L^\frac{q+1}{q}(0,T;W^{-\frac{q+1}{q}})} \\
+ C(\epsilon_1, q + 1)(C^\Omega_{1 (1 + C_F)})^q \|g\|^q_{L^\frac{q+1}{q}(0,T;W^{-\frac{q+1}{q}})} \),
\]

provided that \(\|f - \frac{1}{2}\hat{u}\|_{L^\infty(0,T;L^2(\Omega))} \leq \hat{b} < \min\{\frac{\beta}{2(C^\Omega_{H^1,L^2})^2}, \frac{b(1 - \delta)}{2(C^\Omega_{H^1,L^2})^2}\} = \frac{1}{\frac{4\epsilon_0}{\epsilon_1}}\) and that \(0 < \epsilon_0 < \frac{\eta}{\gamma}\), \(0 < \epsilon_1 < \frac{b\delta}{2}\).

Testing with $\hat{u}$ and adding $\mu$ times the obtained estimate to \([4.45]\) results in a higher order energy estimate valid for arbitrary time:

\[
\mu \left(\frac{\alpha - 1}{2} \|\hat{u}\|^2_{L^2(0,T;L^2(\Omega))} + \mu \left(\frac{b(1 - \delta)}{4} - \sigma\right) \|\nabla \hat{u}\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{b\delta}{2} - \epsilon_0 \right) \|\nabla \hat{u}\|^2_{L^2(0,T;L^2(\Omega))} \\
+ \mu \left(\frac{\alpha}{4} + \beta\right) \|\nabla \hat{u}\|^2_{L^\infty(0,T;L^2(\Omega))} + \mu \left(\frac{b\delta}{2} - \epsilon_0 \right) \|\nabla \hat{u}\|^2_{L^\infty(0,T;L^2(\Omega))} \\
+ \frac{c^2}{4} \left(1 - \mu \frac{c^2}{\sigma}\right) \|\nabla \hat{u}\|^2_{L^\infty(0,T;L^2(\Omega))} + \left(\frac{\beta}{2} - \frac{1}{2} \mu \frac{1}{\sigma}(C^\Omega_{H^1,L^2})^2 \|\hat{u}\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\alpha}{4} \|\hat{u}\|^2_{L^2(0,T;L^2(\Gamma))} \\
\leq C(\Gamma(g) + |u_1|^2_{H^1(\Omega)} + |\nabla u_0|^2_{L^2(\Omega)} + |u_1|^2_{L^2(\Gamma)} + |u_1|^2_{L^2(\Gamma)}),
\]

with $\hat{b} = \frac{b(1 - \delta)}{2} - (C^\Omega_{H^1,L^2})^2 \mu (C^\Omega_{H^1,L^2})^2 \|\hat{u}\|^2_{L^2(0,T;L^2(\Omega))}$, for some appropriately chosen $\overline{C} > 0$. Therefore we obtain:
Proposition 4.7. Let $\beta > 0$ and assumptions (i) of Proposition 4.3 hold, with
\[
\|f - \frac{1}{2} \dot{u}\|_{L^\infty(0,T;L^2(\Omega))} \leq \tilde{b} < \min\left\{\frac{\beta}{2(C^\Omega_{H^1,L^2})^2}, \frac{b(1-\delta)}{2(C^\Omega_{H^1,L^2})^2}\right\}.
\]

Then (4.44) has a unique weak solution in $\tilde{X}$, with $\tilde{X}$ defined as in (4.10), which satisfies (4.45) for some sufficiently small constants $\epsilon_0, \epsilon_1 > 0$.

If, in addition to above, assumptions (ii) in Proposition 4.6 are satisfied, then $u \in X$, with $X$ defined as in (4.32), and $u$ satisfies the energy estimate (4.46) for some sufficiently small constants $\epsilon_0, \epsilon_1, \mu, \sigma, \tau > 0$ and some large enough $C$, independent of $T$.

Addition of a nonlinear lower order damping term. We continue with considering an equation with an added lower order nonlinear damping term:
\[
\begin{align*}
\frac{1}{2} \int_\Omega a\dddot{u}^2 dx + c^2 \nabla u \cdot \nabla (|\dot{u}|^{q-1}) \nabla \dot{u} + \gamma |\dot{u}|^{q-1} \dddot{u} + f \dddot{u} &= 0 \quad \text{in} \quad \Omega \times (0,T], \\
\frac{1}{2} \int_\Omega c^2 \frac{\partial u}{\partial n} + b((1-\delta) + |\nabla \dot{u}|^{q-1}) \frac{\partial u}{\partial n} &= g \quad \text{on} \quad \Gamma \times (0,T], \\
\alpha u + c^2 \frac{\partial u}{\partial n} + b((1-\delta) + |\nabla \dot{u}|^{q-1}) \frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \tilde{\Gamma} \times (0,T], \\
(u, \dddot{u}) &= (u_0, u_1) \quad \text{on} \quad \tilde{\Omega} \times \{t = 0\},
\end{align*}
\]
with $\gamma > 0$, which is motivated by problem (4.6). Once we multiply (4.47) by $\dot{u}$ and integrate by parts, we produce
\[
\begin{align*}
\int_0^t \int_\Omega \frac{a\dddot{u}^2 + c^2 |\nabla u|^2}{2} dx ds + \int_0^t \int_\Omega \left(\frac{1}{2} - \delta\right) |\nabla \dot{u}|^{q-1} |\nabla \dot{u}|^2 dx ds \\
+ \gamma \int_0^t \int_\Omega |\dot{u}|^{q+1} dx ds + \alpha \int_0^t \int_\Gamma |\dot{u}|^2 dx ds \\
= \int_0^t \int_\Omega (f - \frac{1}{2} \dddot{u})(\dot{u})^2 dx ds + \int_0^t \int_\Gamma g \dot{u} dx ds.
\end{align*}
\]
We will make use of the following inequality
\[
\begin{align*}
\int_0^t \int_\Gamma g \dot{u} dx ds \\
\leq \frac{\epsilon_0}{2} ||\dot{u}||^{q+1}_{L^{q+1}(0,T;W^{1,q+1}(\Omega))} + C(q, \alpha, q + 1)(C^\Omega_{H^1,L^2})^{q+1} \frac{L_{q+1}^{q+1}(0,T;W^{1,q+1,\frac{q+1}{q-1}}(\Gamma))^{q+1}}{2},
\end{align*}
\]
and, for $q > 1$,
\[
\begin{align*}
\int_0^t \int_\Omega (f - \frac{1}{2} \dddot{u})(\dot{u})^2 dx ds &\leq \int_0^t \int_\Omega |\dot{u}|^{q+1} dx ds \left(\frac{1}{2} \int_\Omega (f - \frac{1}{2} \dddot{u}) dx \right)^{\frac{q+1}{2}} \left(\int_\Omega |\dot{u}|^{\frac{q+1}{q-1}} dx \right)^{\frac{q-1}{2}} ds \\
&= \int_0^t \int_\Omega |\dot{u}|^{q+1} |f - \frac{1}{2} \dddot{u}|^{\frac{q+1}{q-1}} dx ds \\
\leq \frac{\epsilon_0}{2} ||\dot{u}||^{q+1}_{L^{q+1}(0,T;L^{q+1}(\Omega))} + C(q, \alpha, q+1) \frac{L_{q+1}^{q+1}(0,T;L^{q+1,\frac{q+1}{q-1}}(\Omega))^{q+1}}{2},
\end{align*}
\]
to obtain a lower order energy estimate
\[
\frac{a}{4} ||\dot{u}||^2_{L^\infty(0,T;L^2(\Omega))} + \frac{c^2}{4} ||\nabla u||^2_{L^\infty(0,T;L^2(\Omega))} + \frac{b(1-\delta)}{2} ||\nabla \dot{u}||^2_{L^2(0,T;L^2(\Omega))}.
\]
4.2. Neumann as well as absorbing boundary conditions

\[ + \frac{b \delta - \epsilon_0}{2} \| \nabla \dot{u} \|^q_{L^{q+1}(0,T;L^{q+1}(\Omega))} + \left( \frac{\gamma}{2} - \epsilon_0 \right) \| \dot{u} \|^q_{L^{q+1}(0,T;L^{q+1}(\Omega))} + \frac{\alpha}{2} \| \ddot{u} \|^2_{L^2(\Gamma)} \]

(4.49)

\[ \leq C \left( \frac{\epsilon_0}{2}, q + 1 \right) (C^{tr}_r \| g \|_{L^{q+1}(0,T;W^{-\frac{q+1}{q+1}}(\Gamma))}) \frac{q+1}{q} + \frac{\pi}{2} \| u_1 \|_{L^2(\Omega)} \]

\[ + C \left( \frac{\epsilon_0}{2}, q + 2 \right) \| f - \frac{1}{2} \dot{u} \|^q_{L^{q+1}(0,T;L^{q+1}(\Omega))} + \frac{c^2}{2} | \nabla u_0 |_{L^2(\Omega)} \]

assuming that \( f, \dot{u} \in L^{q+1}(0,T;L^{q+1}(\Omega)) \) and \( 0 < \epsilon_0 < \frac{\gamma}{2} \).

For obtaining a higher order estimate, we multiply (4.47) with \( \ddot{u} \), integrate with respect to space and time and make use of the estimate

\[ \int_0^t \int_\Omega g \ddot{u} \, dx \, ds \leq \eta \| \ddot{u} \|^2_{L^{q+1}(0,T;W^{q+1}(\Omega))} + \frac{\epsilon_0}{2} \| \ddot{u} \|^2_{L^{q+1}(0,T;W^{q+1}(\Omega))} \]

\[ + C(\eta, q + 1) (C^{tr}_r \| g \|_{L^{q+1}(0,T;W^{-\frac{q+1}{q+1}}(\Gamma))}) \frac{q+1}{q} \]

\[ + | u_1 |_{W^{q+1}(\Omega)}^q + C(1, q + 1) (C^{tr}_1 \| g \|_{W^{-\frac{q+1}{q+1}}(\Gamma)}) \frac{q+1}{q} \]

\[ + C \left( \frac{\epsilon_0}{2}, q + 2 \right) \| \ddot{u} \|^q_{L^{q+1}(0,T;L^{q+1}(\Omega))} + \frac{c^2}{2} | \nabla u_0 |_{L^2(\Omega)} \]

(4.50)

In order to avoid dependence on time, we approach estimate (4.37) differently this time: by employing the embedding \( L^{q+1}(\Omega) \hookrightarrow L^4(\Omega) \), valid for \( q \geq 3 \) (see (3.3)), we obtain

\[ \int_0^t \int_\Omega f \ddot{u} \, dx \, ds \leq \frac{1}{2\tau} (C^{tr}_\Omega L^{q+1,L^4})^2 \int_0^t \int_\Omega | f(s) |^2_{L^2(\Omega)} | \dot{u}(s) |^2_{L^{q+1}(\Omega)} \, ds \]

\[ + \frac{\tau}{2} \| \ddot{u} \|^2_{L^2(0,T;L^2(\Omega))} \]

\[ \leq \frac{\epsilon_0}{2} \| \ddot{u} \|^2_{L^{q+1}(0,T;L^{q+1}(\Omega))} + \frac{\tau}{2} \| \ddot{u} \|^2_{L^2(0,T;L^2(\Omega))} \]

\[ + C \left( \frac{\epsilon_0}{2}, q + 2 \right) \left( \frac{1}{2 \tau} (C^{tr}_\Omega L^{q+1,L^4})^2 \| f \|^2_{L^{2(q+1)}(0,T;L^4(\Omega))} \right) \frac{q+1}{q} \]

which, together with (4.49), leads to the higher order estimate

\[ \frac{\mu}{2} \| \ddot{u} \|^2_{L^2(0,T;L^2(\Gamma))} + \frac{c^2}{4} \left( 1 - \frac{\mu^2}{\sigma} \right) \| \nabla u \|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{\alpha}{4} \| \ddot{u} \|^2_{L^2(0,T;L^2(\Omega))} \]

\[ + \mu \left( \frac{b(1 - \delta)}{4} - \sigma \right) \| \nabla \dot{u} \|^2_{L^\infty(0,T;L^2(\Omega))} + \mu \left( \frac{b(1 - \delta)}{2} - \mu^2 \right) \| \nabla \dot{u} \|^2_{L^2(0,T;L^2(\Omega))} \]

\[ + \left( \frac{b\delta - \epsilon_0}{2} (\mu + 1) \right) \| \nabla \ddot{u} \|^2_{L^\infty(0,T;L^2(\Omega))} + \left( \frac{\gamma}{2} - \epsilon_0 (\mu + 1) \right) \| \ddot{u} \|^2_{L^{q+1}(0,T;L^{q+1}(\Omega))} \]

\[ + \mu \left( \frac{b\delta}{2} (q + 1) - \eta \right) \| \nabla \ddot{u} \|^2_{L^\infty(0,T;L^2(\Omega))} + \mu \left( \frac{\gamma}{2} (q + 1) - \eta \right) \| \ddot{u} \|^2_{L^{q+1}(0,T;L^{q+1}(\Omega))} \]

(4.51)

\[ + \mu \left( \frac{\alpha}{4} \| f \|^2_{L^\infty(0,T;L^4(\Gamma))} + \frac{\alpha}{2} | \ddot{u} |^2_{L^2(0,T;L^2(\Gamma))} \right) \]

\[ \leq C \left( \sum_{s=0}^1 \frac{d^s}{dt^s} g \right) \frac{q+1}{q} \| g \|^q_{L^{q+1}(0,T;W^{-\frac{q+1}{q+1}}(\Gamma))} + \| g \|^q_{L^{q+1}(0,T;W^{-\frac{q+1}{q+1}}(\Gamma))} \]

\[ + \| f - \frac{1}{2} \dot{u} \|^q_{L^{q+1}(0,T;L^{q+1}(\Omega))} + \| f \|^q_{L^{2(q+1)}(0,T;L^4(\Omega))} + | u_1 |^2_{H^1(\Omega)} + | \nabla u_0 |_{L^2(\Omega)} \]
Adding of a nonlinear lower order damping term; alternative version. Due to the higher order energy estimate: employing this estimate. Instead, provided assumptions of Propositions 4.3 and 4.6 hold, we then

\[ 0 < q \leq a(x,t) \leq \bar{a}, \]

\[ f \in L^{2(\frac{q+1}{q-1})} \left(0, T; L^4(\Omega)\right), \]

\[ g \in L^{\frac{q+1}{q-1}} \left(0, T; W^{-\frac{q+1}{q-1}, \frac{q+1}{q}}(\Gamma)\right), \]

\[ u_0 \in H^1(\Omega), \quad u_1 \in L^2(\Omega), \]

Then (4.47) has a unique weak solution in \( \tilde{X} \), with \( \tilde{X} \) defined as in (4.10), which satisfies (4.49) for some \( 0 < \epsilon_0 < \frac{\gamma}{2} \).

If, in addition to (i), the following assumptions are satisfied

\[ u \geq \frac{a}{2} \left(1 - \frac{\mu^2}{\sigma} \right) \parallel \nabla u \parallel_{L^\infty(0,T;L^2(\Omega))} + \frac{b}{2} \parallel \nabla \hat{u} \parallel_{L^2(0,T;L^2(\Omega))} \]

\[ + \left(\frac{2}{4} - (C_0^{H, L})^2 bT - \frac{1}{2} \frac{\sigma}{\mu} (C_0^{H, L})^2 b^2 T\right) \parallel \hat{u} \parallel_{L^\infty(0,T;L^2(\Omega))} + \frac{\alpha}{4} \parallel \hat{u} \parallel_{L^2(0,T;L^2(\Omega))} \]

\[ + \left(\frac{b(1 - \delta)}{4} - \sigma\right) \parallel \nabla \hat{u} \parallel_{L^\infty(0,T;L^2(\Omega))} + \left(\frac{b\delta}{2} - \epsilon_0 (\mu + 1)\right) \parallel \nabla \hat{u} \parallel_{L^2(0,T;L^{\gamma+1}(\Gamma))} \]

\[ + \mu \left(\frac{b}{2} - \epsilon_0 (\mu + 1)\right) \parallel \hat{u} \parallel_{L^2(0,T;L^{\gamma+1}(\Omega))} \]

\[ \leq \bar{C} \left( \frac{1}{s_0} \int_a^s \frac{d^s g}{dt} \frac{\parallel \hat{u} \parallel_{L^\infty(0,T;W^{-\frac{q+1}{q}, \frac{q+1}{q}}(\Gamma))} + \parallel g \parallel_{L^\infty(0,T;W^{-\frac{q+1}{q}, \frac{q+1}{q}}(\Gamma))} + \parallel u_1 \parallel_{L^2(\Gamma)} + \parallel u_1 \parallel_{H^1(\Omega)} + \parallel \nabla u_0 \parallel_{L^2(\Omega)} \right), \]

Addition of a nonlinear lower order damping term; alternative version. Due to the terms \( \parallel f - \frac{1}{2} \hat{a} \parallel_{L^{2(\frac{q+1}{q-1})} \left(0, T; L^4(\Omega)\right)} \) and \( \parallel f \parallel_{L^{2(\frac{q+1}{q-1})} \left(0, T; L^4(\Omega)\right)} \) appearing on the right hand side in the estimate (4.51), we will not be able to prove local well-posedness of the problem (4.6) by employing this estimate. Instead, provided assumptions of Propositions 4.3 and 4.6 hold, we could proceed with the same estimates as in the proofs of those propositions, with the exception of applying (4.48) and (4.50) to evaluate boundary integrals. This would result in the following higher order energy estimate:

\[ \mu \left(\frac{b}{2} - \epsilon_0 (\mu + 1)\right) \parallel \hat{u} \parallel_{L^2(0,T;L^{\gamma+1}(\Omega))} \]

\[ + \left(\frac{b}{2} - \epsilon_0 (\mu + 1)\right) \parallel \hat{u} \parallel_{L^2(0,T;L^{\gamma+1}(\Omega))} \]

\[ \leq \bar{C} \left( \frac{1}{s_0} \int_a^s \frac{d^s g}{dt} \frac{\parallel \hat{u} \parallel_{L^\infty(0,T;W^{-\frac{q+1}{q}, \frac{q+1}{q}}(\Gamma))} + \parallel g \parallel_{L^\infty(0,T;W^{-\frac{q+1}{q}, \frac{q+1}{q}}(\Gamma))} + \parallel u_1 \parallel_{L^2(\Gamma)} + \parallel u_1 \parallel_{H^1(\Omega)} + \parallel \nabla u_0 \parallel_{L^2(\Omega)} \right), \]
with \( \tilde{b} = \frac{b(1-\delta)}{2} - (C_{H^1,L^2})^2 b - \mu(\frac{1}{2\pi} (C_{H^1,L^4})^2 b^2 + c^2) \), for some appropriately chosen constants \( \tau, \sigma, \epsilon_0, \eta, \mu > 0 \) and large enough \( C \), independent of \( T \). Note, however, that \( T \) appears in the left hand side of the estimate.

**Proposition 4.9.** Let \( \gamma > 0 \) and let the assumptions of Proposition 4.3 hold. Furthermore, let assumptions (ii) in Proposition 4.6 be satisfied. Then problem (4.4.7) has a unique weak solution in \( \tilde{X} \), with \( \tilde{X} \) defined as in (4.10), which satisfies the estimate (4.52) for some sufficiently small constants \( \epsilon_0, \epsilon_1, \mu, \tau, \sigma > 0 \) and some large enough \( \tilde{C} \).

### 4.2. Neumann as well as absorbing boundary conditions

We can now prove local well-posedness for the initial-boundary value problem (4.7). By relying on Proposition 4.6, we find that there exists a unique weak solution \( u \in W(\Omega) \) such that for all \( (0, T) \), \( u \) solves (4.13) with

\[
C_{\Gamma}(g) + |v_0|^2_{L^1(\Omega)} + |\nabla v_0|^2_{L^{q+1}(\Omega)} + |u_1|^2_{H^1(\Omega)} + |\nabla u_0|^2_{L^2(\Omega)} + |u_1|^q_{L^2(\Omega)} \\
+ |u_1|^2_{L^2(\hat{\Gamma})} \leq \kappa_T^2
\]

there exists a unique weak solution \( u \in \mathcal{W} \) of (4.7), where

\[
\mathcal{W} = \{ v \in X : \|v\|_{L^2(0,T;L^2(\Omega))} \leq \tilde{m} \wedge \|v\|_{L^\infty(0,T;H^1(\Omega))} \leq \tilde{m} \wedge \|v\|_{L^{q+1}(0,T;L^{q+1}(\Omega))} \leq \tilde{M} \wedge (v,v)_{L^2} = 0 = (u_0, u_1) \},
\]

with

\[
2|k|C_{W^{1,q+1,L}}^0 \left[ \max\{1 + C_P, C_{1}^0 \} \kappa_T + (1 + C_P)T \frac{q}{q+1} \tilde{M} + C_{2}^0 T \tilde{m} \right] < 1,
\]

and \( \tilde{m} \) and \( \tilde{M} \) sufficiently small, where \( C_{\Gamma}(g) \) is defined as in (4.34).

**Remark 4.11.** In the case of \( \hat{\Gamma} = \emptyset \), the theorem holds with \(|u_1|^2_{L^2(\hat{\Gamma})} \) omitted from the condition (4.53).

**Proof.** We will carry out the proof by using a fixed point argument, i.e. Theorem 3.11. We define an operator \( T : \mathcal{W} \to X, v \mapsto Tv = u \), where \( u \) solves (4.13) with

\[
a = 1 - 2kv, \quad f = -2kv.
\]

We will show that assumptions of Proposition 4.6 are satisfied. Since \( v \in \mathcal{W} \), and \( q > d - 1 \) so we can make use of the embedding \( W^{1,q+1}(\Omega) \hookrightarrow L^\infty(\Omega) \), we have by (3.9)

\[
|2kv(x,t)| \leq 2|k|C_{W^{1,q+1,L}}^0 \left[ \max\{1 + C_P, C_{1}^0 \} \kappa_T + (1 + C_P)T \frac{q}{q+1} \tilde{M} + C_{2}^0 T \tilde{m} \right],
\]

and \( \dot{a} = -2kv \in L^\infty(0,T;L^2(\Omega)). \)

It follows that 0 < \( \frac{a}{\tilde{\alpha}} = 1 - a_1 < a < \bar{\alpha} = 1 + a_1 \), where

\[
a_1 := 2|k|C_{W^{1,q+1,L}}^0 \left[ \max\{1 + C_P, C_{1}^0 \} \kappa_T + (1 + C_P)T \frac{q}{q+1} \tilde{M} + C_{2}^0 T \tilde{m} \right].
\]
Chapter 4. Local well-posedness results

Furthermore,
\[
\|f - \frac{1}{2} \hat{u}\|_{L^\infty(0,T;L^2(\Omega))} = \|k\hat{v}\|_{L^\infty(0,T;L^2(\Omega))} \leq |k|\bar{m},
\]
\[
\|f\|_{L^\infty(0,T;L^1(\Omega))} = 2|k|\|\hat{v}\|_{L^\infty(0,T;L^1(\Omega))} \leq 2|k|C_{H^1,L^1}^2 \bar{m} = \bar{b}.
\]

Hence the higher order energy estimate (4.33) holds and by choosing \(\bar{m}, \bar{M} > 0\) such that
\[
2|k|C_{W^1,q+1,L^\infty}^0 ((1 + C_P)T^{\frac{q}{2}} \bar{M} + C_2^0 T \bar{m}) < 1,
\]
\[
\bar{m} < \frac{1}{|k|} \min \left\{ \frac{b(1 - \delta)}{2|C_{W^1,L^1}^0|^2}, \frac{a}{4T(C_{W^1,L^1}^0)^2} \right\},
\]
and making the bound \(\kappa_T\) on initial and boundary data small enough
\[
\kappa_T < \frac{1}{\max\{1 + C_P, C_1^0\}} \left( \frac{1}{2|C_{W^1,q+1,L^\infty}^0|} - (1 + C_P)T^{\frac{q}{2}} \bar{M} - C_2^0 T \bar{m} \right),
\]
\[
\kappa_T^2 \leq \frac{1}{\min (\frac{a}{2}, - \frac{\tau}{2})} \min \left\{ \left( \frac{a}{4} - (C_{W^1,L^1}^0)^2 \right) 2\bar{M} - \epsilon_0(\mu + 1) - \frac{1}{2\tau} (C_{W^1,L^1}^0)^4 \bar{b}^2 T \bar{m}^2, \right. \]
\[
\left. \left( \frac{\alpha - \tau}{2} \bar{m}^2, \frac{\mu (b(1 - \delta))}{4} - \sigma \right) \bar{m}^2, \left( \frac{b\delta}{2} - \epsilon_1(\mu + 1) \right) \bar{m}^{\frac{q+1}{2}} \right\},
\]
we achieve that \(u \in \mathcal{W}\), with constants \(\epsilon_0, \epsilon_1, \tau, \eta, \sigma, \mu\) chosen as in (4.12) and (4.39) and \(\mathcal{C}\) as in (4.33).

In order to prove contractivity, consider \(v_i \in \mathcal{W}, u_i = T v_i \in \mathcal{W}, i = 1, 2\) and denote \(\hat{u} = u_1 - u_2, \hat{v} = v_1 - v_2\). Subtracting equation (4.8) for \(u^1\) and \(u^2\) yields:
\[
\begin{cases}
(1 - 2k_v)\hat{u} - c^2 \Delta \hat{u} - b(1 - \delta) \Delta \hat{u} - b\delta \text{div}(|\nabla \hat{u}_1|^{q-1} \nabla \hat{u}_1 - |\nabla \hat{u}_2|^{q-1} \nabla \hat{u}_2),
\end{cases}
\]
\[
= 2k(\hat{v}_2 + \hat{v}_1 \hat{u} + \hat{v}_2 \hat{u}) \text{ in } \Omega,
\]
\[
c^2 \frac{\partial \hat{u}}{\partial t} + b(1 - \delta) \frac{\partial \hat{u}}{\partial t} + b\delta(|\nabla \hat{u}_1|^{q-1} \frac{\partial \hat{u}_1}{\partial t} - |\nabla \hat{u}_2|^{q-1} \frac{\partial \hat{u}_2}{\partial t}) = 0 \text{ on } \Gamma,
\]
\[
\alpha \hat{u} + c^2 \frac{\partial \hat{u}}{\partial t} + b(1 - \delta) \frac{\partial \hat{u}}{\partial t} + b\delta(|\nabla \hat{u}_1|^{q-1} \frac{\partial \hat{u}_1}{\partial t} - |\nabla \hat{u}_2|^{q-1} \frac{\partial \hat{u}_2}{\partial t}) = 0 \text{ on } \hat{\Gamma},
\]
\[
(\hat{u}, \hat{\hat{u}})|_{t=0} = (0, 0).
\]

After testing (4.57) with \(\hat{u}\) and making use of inequality (4.29), we obtain
\[
\frac{1}{2} \left[ \int_{0}^{t} (1 - 2k_v)(\hat{u})^2 dx + c^2 |\nabla \hat{u}|_{L^2(\Omega)}^2 \right]_0^t + b(1 - \delta) \int_{0}^{t} \left| \nabla \hat{u} \right|_{L^2(\Omega)}^2 ds + \alpha \int_{0}^{t} \left| \hat{u} \right|_{L^2(\Omega)}^2 ds
\]
\[
\leq 2|k| \int_{0}^{t} \int_{\Omega} \left( \frac{1}{2} v_1(\hat{u})^2 + \hat{v}_2 \hat{u} + \hat{v}_2 \hat{u} \right) dx ds,
\]
and therefore we have
\[
\frac{1}{2} \left[ \int_{0}^{t} (1 - 2k_v)(\hat{u})^2 dx + c^2 |\nabla \hat{u}|_{L^2(\Omega)}^2 \right]_0^t + b(1 - \delta) \int_{0}^{t} \left| \nabla \hat{u} \right|_{L^2(\Omega)}^2 ds + \alpha \int_{0}^{t} \left| \hat{u} \right|_{L^2(\Omega)}^2 ds
\]
\[
\leq |k|(C_{H^1,L^1}^0)^2 \left( \|v_1\|_{L^\infty(0,T;L^2(\Omega))} \int_{0}^{t} \left| \hat{u} \right|_{H^1(\Omega)}^2 ds + \|v_2\|_{L^2(0,T;L^2(\Omega))} \|\hat{u}_2\|_{L^2(0,T;H^1(\Omega))} \right)
\]
\[
+ \int_{0}^{t} \left| \hat{u} \right|_{H^1(\Omega)}^2 ds + \|v_2\|_{L^\infty(0,T;L^2(\Omega))} \|\hat{u}_2\|_{L^2(0,T;H^1(\Omega))} + \int_{0}^{t} \left| \hat{u} \right|_{H^1(\Omega)}^2 ds \right).
\]
Utilizing the fact that $v_1, v_2, u_1, u_2 \in \mathcal{W}$ and the inequalities

$$\|\nabla \hat{v}\|_{L^\infty(0,T,L^2(\Omega))} \leq T \|\nabla \hat{v}\|_{L^2(0,T,L^2(\Omega))},$$

and

$$\|\hat{v}\|_{L^\infty(0,T,L^2(\Omega))} \leq T \|\hat{v}\|_{L^2(0,T,L^2(\Omega))},$$

leads to the estimate

$$\frac{1}{4} - \frac{a_1}{4} \|\hat{u}\|_{L^\infty(0,T,L^2(\Omega))}^2 + \frac{c_2^2}{4} \|\nabla \hat{u}\|_{L^\infty(0,T,L^2(\Omega))}^2 + \frac{b(1-\delta)}{2} \|\nabla \hat{u}\|_{L^2(0,T,L^2(\Omega))}^2 \leq \frac{k}{|C^2_{H^1,L^4}|} \bar{m}(T+1) \max\{1,T\} \left( \|\hat{v}\|_{L^\infty(0,T,L^2(\Omega))}^2 + \|\nabla \hat{v}\|_{L^2(0,T,L^2(\Omega))}^2 \right),$$

and altogether we have

$$\min\left\{ \frac{1}{4} - \frac{a_1}{4} - 3T \frac{k}{|C^2_{H^1,L^4}|} \bar{m}^2, \frac{b(1-\delta)}{2} - 3 \frac{k}{|C^2_{H^1,L^4}|} \bar{m}^2 \right\} \|u\|^2 \leq \frac{k}{|C^2_{H^1,L^4}|} \bar{m}(T+1) \max\{1,T\} \|v\|^2,$$

where $\|u\|^2 = \|\hat{u}\|_{L^\infty(0,T,L^2(\Omega))}^2 + \|\nabla \hat{u}\|_{L^2(0,T,L^2(\Omega))}^2 + \|\nabla \hat{u}\|_{L^\infty(0,T,L^2(\Omega))}^2$. We conclude from (4.58) that $T$ is a contraction with respect to the norm $\|\cdot\||$, provided that $\bar{m}$ is sufficiently small. This, together with the self-mapping property and $\mathcal{W}$ being closed, provides us with existence and uniqueness of a solution.

Addition of a linear lower order damping term. By relying on Proposition 4.7, we can obtain local well-posedness of the problem (4.5) with $\beta > 0$. Since we need to avoid degeneracy of the term $1 - 2ku$ and therefore make use of the estimate (3.9) so that the condition (4.55) is satisfied, we cannot completely avoid restriction on the final time in the case of the fully nonlinear equation. Inspecting the proof of Theorem 4.10 immediately yields:

**Theorem 4.12.** Let $\beta > 0$ and the assumptions of Theorem 4.10 hold. For any $T > 0$ there is a $\kappa_T > 0$ such that for all $u_0, u_1 \in W^{1,q+1}(\Omega)$, with (4.53), there exists a unique weak solution $u \in \mathcal{W}$ of (4.5), where $\mathcal{W}$ is defined as in (4.54), with (4.55) and $\bar{m}$ and $M$ sufficiently small.

Addition of a nonlinear lower order damping term. For obtaining well-posedness of the problem (4.10) with $\gamma > 0$, we cannot rely on estimates in Proposition 4.8 to prove self-mapping of the fixed-point operator $T$, instead we make use of (4.52) and Proposition 4.9; therefore restrictions on the final time persist in the case of the fully nonlinear equation. Analogously to Theorem 4.10 we obtain:
Theorem 4.13. Let $\gamma > 0$ and the assumptions of Theorem 4.10 hold. For any $T > 0$ there is a $\kappa_T > 0$ such that for all $u_0, u_1 \in W^{1, q+1}(\Omega)$, with

$$
\begin{align*}
\sum_{s=0}^1 \| \frac{d^s}{dt^s} g \|_{L^{q+1}(0, T; W^{-\frac{q}{q+s}, \frac{q+1}{q+s}}(\Gamma))} + & \| g \|_{L^{q+1}(0, T; W^{-\frac{q}{q+s}, \frac{q+1}{q+s}}(\Gamma))} + |u_0|_{L^2(\Omega)}^2 \\
+ |\nabla u_0|_{L^{q+1}(\Omega)}^2 + |u_1|_{H^1(\Omega)}^2 + |\nabla u_0|_{L^2(\Omega)}^2 + |u_1|_{W^{1,q+1}(\Omega)}^2 + |u_1|_{L^2(\Gamma)}^2 & \leq \kappa_T^2,
\end{align*}
$$

(4.59)

there exists a unique weak solution $u \in \mathcal{W}$ of (4.6), where $\mathcal{W}$ is defined as in (4.54), with (4.55), and $\bar{m}$ and $\bar{M}$ sufficiently small.

4.3 WELL-POSEDNESS FOR THE ACOUSTIC-Acoustic COUPLING PROBLEM

We will now consider the problem of an acoustic-acoustic coupling, modeled by the weak form of (2.4) under appropriate boundary conditions and with coefficients varying in space. We will make the following assumptions on coefficients:

$$
\begin{align*}
\lambda, k, q, b, \delta & \in L^\infty(\Omega), \alpha \in L^\infty(\Gamma) \\
\bar{w} & := |w|_{L^\infty(\Omega)} \text{ for } w \in \{b, q, \lambda, \delta, k\}, \bar{\delta} < 1, \\
\exists \bar{q}, b, \bar{\delta}, \bar{\alpha} : \lambda & \geq \bar{\lambda} > 0, \bar{q} \geq \bar{q} > 0, b \geq \bar{b} > 0, \bar{\delta} \geq \bar{\delta} > 0, \bar{\alpha} \geq \bar{\alpha} > 0.
\end{align*}
$$

(4.60)

**Dirichlet boundary conditions.** Let us first recall the well-posedness result for the following Dirichlet problem:

$$
\begin{align*}
\text{Find } u \text{ such that } \\
\int_0^T \int_{\Omega} \left\{ \frac{1}{\lambda(x)} (1 - 2k(x)u) \hat{u} \phi + \frac{1}{\bar{q}(x)} \nabla u \cdot \nabla \phi + b(x)(1 - \delta(x)) \nabla \hat{u} \cdot \nabla \phi \\
+ b(x)\delta(x)|\nabla \hat{u}|^{q-1} \nabla \hat{u} \cdot \nabla \phi - \frac{2k(x)}{\lambda(x)} (\hat{u})^2 \phi \right\} dx \, ds = 0
\end{align*}
$$

(4.61)

holds for all test functions $\phi \in \tilde{X}$, with $(u, \hat{u})_{t=0} = (u_0, u_1)$. $\tilde{X} = L^2(0, T; W_0^{1,q+1}(\Omega))$. The strong form then reads as:

$$
\begin{align*}
\frac{1}{\lambda(x)} (1 - 2k(x)u) \hat{u} - \text{div} \left( \frac{1}{\bar{q}(x)} \nabla u \right) - \text{div} \left( b(x)((1 - \delta(x)) + \delta(x)|\nabla \hat{u}|^{q-1}) \nabla \hat{u} \right) \\
= \frac{2k(x)}{\lambda(x)} (\hat{u})^2 \text{ in } \Omega \times (0, T], \\
u = 0 \text{ on } \partial \Omega \times (0, T], \\
(u, \hat{u}) = (u_0, u_1) \text{ on } \overline{\Omega} \times \{t = 0\}.
\end{align*}
$$

(4.62)

**Proposition 4.14.** [9], Theorem 2.3 Let $q > d - 1$, $q \geq 1$ and the assumptions (4.60) on coefficients hold. For any $T > 0$ there is a $\kappa_T > 0$ such that for all $u_0, u_1 \in W^{1, q+1}(\Omega)$ with

$$
|u_1|_{L^2(\Omega)}^2 + |\nabla u_0|_{L^2(\Omega)}^2 + |\nabla u_1|_{L^2(\Omega)}^2 + |\nabla u_0|_{L^{q+1}(\Omega)}^2 + |\nabla u_1|_{L^{q+1}(\Omega)}^2 \leq \kappa_T^2
$$

there exists a unique solution $u \in \mathcal{W} \subset X = H^2(0, T; L^2(\Omega)) \cap W^{1,\infty}(0, T; W_0^{1,q+1}(\Omega))$ of (4.61), and

$$
\mathcal{W} = \{v \in X : \| \hat{v} \|_{L^2(0,T;L^2(\Omega))} \leq \bar{m} \wedge \| \nabla \hat{v} \|_{L^\infty(0,T;L^2(\Omega))} \leq \bar{M} \}.
$$

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4.3. Well-posedness for the acoustic-acoustic coupling problem

\[ \land \| \nabla \hat{v} \|_{L^{q+1}(0,T;L^{q+1}(\Omega))} \leq \hat{M} \land (v, \hat{v}) = (u_0, u_1), \]

with

\[ \tilde{E} C_{W_0^{1,q+1,L^{\infty}}}^{\Omega} (\kappa_T + T^{\frac{q}{q+1}} \hat{M}) < 1, \]

and \( \tilde{m} \) sufficiently small. Furthermore, the following estimate holds:

\begin{align}
\| u \|_{L^\infty(0,T;L^2(\Omega))}^2 + \| \hat{u} \|_{L^2(0,T;L^2(\Omega))}^2 + \| \nabla \hat{u} \|_{L^{q+1}(0,T;L^{q+1}(\Omega))}^2 \\
+ \| \nabla u \|_{L^\infty(0,T;L^2(\Omega))}^2 + \| \nabla \hat{u} \|_{L^{q+1}(0,T;L^{q+1}(\Omega))}^2 \\
\leq C (\| u_1 \|_{L^2(\Omega)}^2 + \| \nabla u_0 \|_{L^2(\Omega)}^2 + \| \nabla u_1 \|_{L^{q+1}(\Omega)}^2),
\end{align}

where \( C \) depends on \( \lambda, \varrho, b, \delta, k \), and the norm \( C_{W_0^{1,q+1,L^1}}^{\Omega} \) of the embedding \( H_0^1(\Omega) \hookrightarrow L^1(\Omega) \).

**Neumann and absorbing boundary conditions.** Next, we would like to investigate the acoustic-acoustic coupling under Neumann as well as absorbing boundary data:

\begin{align}
\begin{cases}
\frac{1}{\lambda(x)} (1 - 2k(x)u) \hat{u} - \text{div} \left( \frac{1}{\varrho(x)} \nabla u \right) - \text{div} (b(x) ((1 - \delta(x)) + \delta(x) |\nabla \hat{u}|^{q-1}) \nabla \hat{u}) \\
= 2k(x) (\hat{u})^2 \quad \text{in } \Omega \times (0,T],
\end{cases}
\end{align}

\begin{align}
\begin{cases}
\frac{1}{\varrho(x)} \frac{\partial u}{\partial n} + b(x) ((1 - \delta(x)) + \delta(x) |\nabla \hat{u}|^{q-1}) \frac{\partial \hat{u}}{\partial n} = g \quad \text{on } \Gamma \times (0,T],
\end{cases}
\end{align}

under the same assumptions (4.60) on the coefficients.

We can again first inspect the problem with nonlinearity present only in the damping term:

\begin{align}
\begin{cases}
a \hat{u} - \text{div} \left( \frac{1}{\varrho(x)} \nabla u \right) - \text{div} (b(x) ((1 - \delta(x)) + \delta(x) |\nabla \hat{u}|^{q-1}) \nabla \hat{u}) \\
+ f \hat{u} = 0 \quad \text{in } \Omega \times (0,T],
\end{cases}
\end{align}

\begin{align}
\begin{cases}
\frac{1}{\varrho(x)} \frac{\partial u}{\partial n} + b(x) ((1 - \delta(x)) + \delta(x) |\nabla \hat{u}|^{q-1}) \frac{\partial \hat{u}}{\partial n} = g \quad \text{on } \Gamma \times (0,T],
\end{cases}
\end{align}

\begin{align}
\begin{cases}
\alpha(x) \hat{u} + \frac{1}{\varrho(x)} \frac{\partial u}{\partial n} + b(x) ((1 - \delta(x)) + \delta(x) |\nabla \hat{u}|^{q-1}) \frac{\partial \hat{u}}{\partial n} = 0 \quad \text{on } \Gamma \times (0,T],
\end{cases}
\end{align}

\begin{align}
\begin{cases}
(u, \hat{u}) = (u_0, u_1) \quad \text{on } \Omega \times \{ t = 0 \}.
\end{cases}
\end{align}

Analogously to Proposition 4.3, Proposition 4.6, and Theorem 4.10, we obtain:

**Corollary 4.15.** Let assumptions (4.60) and assumptions (i) in Proposition 4.3 be satisfied, with

\[ \| f - \frac{1}{4} \hat{u} \|_{L^\infty(0,T;L^2(\Omega))} \leq \hat{b} < \min \left\{ \frac{b(1 - \delta)}{2 (C_{H^1,L^2})^2}, \frac{a}{4T (C_{H^1,L^2})^2} \right\}. \]

Then (4.65) has a unique weak solution \( u \in \hat{X} \), with \( \hat{X} \) defined as in (4.10), which satisfies the energy estimate

\[ \left[ \frac{\alpha}{4} - \hat{b} (C_{H^1,L^2})^2 T - \epsilon_0 \right] \| u \|_{L^\infty(0,T;L^2(\Omega))}^2 + \frac{\epsilon_0^2}{4} \| \nabla u \|_{L^\infty(0,T;L^2(\Omega))}^2 + \frac{\alpha}{2} \| \hat{u} \|_{L^2(0,T;L^2(\Gamma))}^2 \]
\[ + \left[ \frac{b(1 - \delta)}{2} - b(C_{H^1,L^2})^2 \right] \| \nabla \dot{u} \|_{L^2(0,T;L^2(\Omega))}^2 + \left[ \frac{b\delta}{2} - \epsilon_1 \right] \| \nabla \dot{u} \|_{L^{q+1}(0,T;L^{q+1}(\Omega))}^{q+1} \]

\[ \leq \frac{\pi}{2} |u_1|_{L^2(\Omega)}^2 + \frac{\pi^2}{2} |\nabla u_0|_{L^2(\Omega)}^2 + \frac{1}{4\epsilon_0} (C_1 C_2^\delta)^2 \| g \|_{L^1(0,T;W^{-\frac{q}{2}+\frac{q+1}{4}}(\Gamma))}^2 \]

\[ + C(\epsilon_1, q + 1)(C_1^\delta (1 + C_P))^{\frac{q+1}{2}} \| g \|_{L^\frac{q}{2}+\frac{q+1}{4}(0,T;W^{-\frac{q}{2}+\frac{q+1}{4}}(\Gamma))}^\frac{q+1}{2} \]

for some sufficiently small constants \( \epsilon_0, \epsilon_1 > 0 \).

If, additionally, assumptions (ii) of Proposition 4.6 hold, then \( u \in X \), where \( X \) is defined as in (4.32), and satisfies the energy estimate (4.33), where \( \alpha \) is replaced with \( \alpha_1 \), \( b(1 - \delta) \) with \( b(1 - \delta) \), \( b\delta \) with \( b\delta \) and \( \frac{c^2}{4}(1 - \mu \frac{c^2}{4}) \) with \( \frac{c^2}{4} - \mu \frac{c^2}{4 \sigma} \), for some small enough constants \( \tau, \eta, \sigma, \mu > 0 \) and some large enough \( C \), independent of \( T \).

**Corollary 4.16.** (Local well-posedness for the problem of acoustic-acoustic coupling) Let \( g \in L^\infty(0,T;W^{-\frac{q}{2}+\frac{q+1}{4}}(\Gamma)) \), \( \dot{g} \in L^{q+1}(0,T;W^{-\frac{q}{2}+\frac{q+1}{4}}(\Gamma)) \), \( q > d - 1, q \geq 1 \) and assumptions (4.60) be satisfied. For any \( T > 0 \) there is a \( \kappa_T > 0 \) such that for all \( u_0, u_1 \in W^{1,q+1}(\Omega) \), with (4.53), there exists a unique weak solution \( u \in W \) of (4.64), where \( W \) is defined as in (4.54), with (4.55) with \( |k| \) replaced by \( |k|_{L^\infty(\Omega)} \), and \( \bar{m} \) and \( M \) sufficiently small. Moreover, \( u \) continuously depends on the initial and boundary data in the sense of (4.31).
CHAPTER 5

Higher regularity for the Westervelt equation with strong nonlinear damping

In this chapter, we will obtain higher interior regularity results for solutions of the Westervelt equation with strong nonlinear damping (2.4). We will show that
\[ u \in H^1(0,T; H^2_{\text{loc}}(\Omega)) \text{ and } \|
abla \dot{u}\|_{L^q}^{\frac{q-1}{2}} \nabla \dot{u} \in L^2(0,T; H^1_{\text{loc}}(\Omega)). \]

Secondly, we will consider the coupled problem and show that its solution is piecewise $H^2$-regular in space, i.e. $H^2$-regular up to the boundary on each of the subdomains, under the assumption that the gradient of the acoustic pressure remains essentially bounded in space and time. This result is crucial in future numerical approximations of the present problem, as well as in the forthcoming shape sensitivity analysis for finding the optimal shape of the focusing acoustic lens, where $H^2$-regularity of $u$ will be needed in order to express the shape derivative in terms of integrals over the boundary of the lens.

5.1 Difference quotients

In what is to follow, we will need to employ difference quotient approximations to weak derivatives. Assume that $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$ is an open, connected set with Lipschitz boundary. Let $V \subset \subset \Omega$. $D^l_r$ will stand for the $r$-th difference quotient of size $l$
\[ D^l_r u(x,t) = \frac{u(x + le_r, t) - u(x, t)}{l}, \quad r \in [1,d], \]
for $x \in V$, $l \in \mathbb{R}$, $0 < |l| < \frac{1}{2}\text{dist}(V, \partial \Omega)$. Then $D^l u := (D^l_1 u, \ldots, D^l_d u)$. We recall the integration by parts formula for difference quotients
\[ \int_V u D^l_r \varphi \, dx = - \int_V D^{-l}_r u \varphi \, dx, \]
where $\varphi \in C_c^\infty(V)$, $0 < |l| < \frac{1}{2}\text{dist}(V, \partial \Omega)$, as well as the product rule
\[ D^l_r(\varphi u) = \varphi^l D^l_r u + u D^l_r \varphi, \]
with
\[ \varphi^l(x,t) := \varphi(x + le_r, t). \]

We will also need the following result:

**Lemma 5.1.** [20], Theorem 3, Section 5] (a) Assume $0 \leq q < \infty$ and $u \in W^{1,q+1}(\Omega)$. Then for each $V \subset \subset \Omega$
\[ \|D^l u\|_{L^{q+1}(V)} \leq C \|
abla u\|_{L^{q+1}(\Omega)}, \]
for some constant $C$ and all $0 < |l| < \frac{1}{2}\text{dist}(V, \partial \Omega)$.

(b) Assume $0 < q < \infty$, $u \in L^{q+1}(\Omega)$, and there exists a constant $C$ such that $\|D^l u\|_{L^{q+1}(V)} \leq C$ for all $0 < |l| < \frac{1}{2}\text{dist}(V, \partial \Omega)$. Then
\[
u \in W^{1,q+1}(V), \text{ with } \|\nabla u\|_{L^{q+1}(V)} \leq C.
\]

### 5.2 Higher interior regularity

In this section, we will establish higher interior regularity for the equation (2.4) with constant coefficients. Let us first consider the Dirichlet problem (4.1) with the assumptions (4.2). Choose any open set $\Omega$ and an open set $W \subset \subset \Omega$. We then introduce a smooth cut-off function $\chi$ such that
\[
\begin{cases}
\chi = 1 \text{ on } V, & \chi = 0 \text{ on } \Omega \setminus W, \\
0 \leq \chi \leq 1.
\end{cases}
\]

Let $|l| > 0$ be small and choose $r \in \{1, \ldots, d\}$. We are then allowed to use
\[
\phi := -D_r^{-1}(\zeta D_r^l \hat{u})\chi_{[0,t]}, \quad t \in [0,T]
\]
as a test function in (4.3), which results in
\[
\begin{aligned}
\frac{1}{2}\left[\int_\Omega (1 - 2kD_r^l \hat{u})^2 dx \right]^t_0 + & \frac{1}{2}c^2 \left[\int_\Omega |\nabla u|^2 dx \right]^t_0 \\
+ b(1 - \delta) & \int_0^t \int_\Omega |\nabla \dot{u} |^2 dx ds + \frac{4}{(q + 1)^2} b\delta \int_0^t \int_\Omega |\nabla D_r^l F|^2 dx ds
\end{aligned}
\]
(5.2)
\[
\begin{aligned}
& \leq 2k \int_0^t \int_\Omega D_r^l \hat{u} \chi \zeta D_r^l \hat{u} dx ds - c^2 \int_0^t \int_\Omega \zeta \nabla \zeta \cdot D_r^l \nabla u D_r^l \hat{u} dx ds \\
& - 2b(1 - \delta) \int_0^t \int_\Omega \zeta \nabla \zeta \cdot D_r^l \nabla \hat{u} D_r^l \hat{u} dx ds \\
& - 2b\delta \int_0^t \int_\Omega D_r^l (|\nabla \hat{u}|^{q-1} \nabla \hat{u}) \cdot |\nabla \zeta| D_r^l \hat{u} dx ds + k \int_0^t \int_\Omega (\hat{u}^l + 2\hat{u})(|\nabla D_r^l \hat{u}|^2) dx ds.
\end{aligned}
\]

Here we have made use of the estimate
\[
\int_0^t \int_\Omega \zeta^2 D_r^l (|\nabla \hat{u}|^{q-1} \nabla \hat{u}) D_r^l \nabla \hat{u} dx ds \geq \frac{4}{(q + 1)^2} \int_0^t \int_\Omega \frac{1}{2} \zeta^2 |\nabla \hat{u} |^{q-1} \nabla \hat{u} - |\nabla \hat{u} |^{q-1} \nabla \hat{u} |^2 dx ds,
\]

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5.2. Higher interior regularity

which follows from \(3.13\). Next, we estimate the terms on the right hand side containing \(\zeta \nabla \zeta\). We have

\[
- c^2 \int_0^t \int_\Omega \zeta \nabla \zeta \cdot D_t^f \nabla u \, D_t^f \dot{u} \, dx
ds - 2b(1 - \delta) \int_0^t \int_\Omega \zeta \nabla \zeta \cdot D_t^f \nabla \dot{u} \, D_t^f \dot{u} \, dx \, ds
\]

\[
\leq C \int_0^T \int_\Omega \zeta(|D_t^f \nabla u| + |D_t^f \nabla \dot{u}|) \, |\nabla \zeta| \, D_t^f \dot{u} \, dx \, ds
\]

\[
\leq \varepsilon T \|\zeta D_t^f \nabla u\|_{L^2(0,T;L^2(\Omega))}^2 + \|\zeta D_t^f \nabla \dot{u}\|_{L^2(0,T;L^2(\Omega))}^2 + \frac{C}{\varepsilon}\|\nabla \dot{u}\|_{L^2(0,T;L^2(\Omega))}^2,
\]

where \(C\) depends on \(c, b, \delta\) and \(|\nabla \zeta|_{L^\infty(W)}\) and we have used Lemma 5.1 (a). By employing estimate \(3.13\) and Hölder’s inequality we obtain

\[
- 2b\delta \int_0^t \int_\Omega |D_t^f| \nabla u|^{q-1} \nabla u| \cdot \zeta \nabla \zeta D_t^f \dot{u} \, dx \, ds
\]

\[
\leq 2b\delta \int_0^T \int_\Omega \frac{1}{2} |D_t^f| \nabla u|^{q-1} \nabla u| - |\nabla \dot{u}|^{q-1} \nabla u| \zeta |\nabla \zeta| \, dx \, ds
\]

\[
\leq 2qb\delta \int_0^T \int_\Omega \nabla \zeta D_t^f \dot{u}|(|\nabla \dot{u}|^{\frac{q-1}{q}} + |\nabla u|^{\frac{q-1}{q}})|D_t^f | \, F| \, dx \, ds
\]

\[
\leq 2qb\delta \left\{ \int_0^T \int_\Omega \nabla \zeta D_t^f \dot{u}|^{q+1} \, dx \, ds \right\}^\frac{q-1}{2(q+1)} \left\{ \int_0^T \int_\Omega \zeta D_t^f | \, F|^{q+1} \, dx \, ds \right\}^\frac{1}{2(q+1)}
\]

\[
\times \left\{ \int_0^T \int_\Omega |\nabla \dot{u}|^{q+1} \, dx \, ds \right\}^\frac{1}{2(q+1)}
\]

The second integral can be majorized with the help of Minkowski’s inequality by

\[
\leq \left\{ \int_0^T \int_\Omega \zeta D_t^f \dot{u}|^{q+1} \, dx \, ds \right\}^\frac{q-1}{2(q+1)} \leq 2\|\nabla \dot{u}\|_{L^{q+1}(0,T;L^{q+1}(\Omega))}^{\frac{q-1}{2(q+1)}},
\]

for small \(|l|\). Utilizing Young’s inequality \(3.10\) then yields

\[
- 2b\delta \int_0^t \int_\Omega \nabla \zeta \cdot D_t^f \nabla \dot{u} \cdot \zeta \nabla \zeta D_t^f \dot{u} \, dx \, ds
\]

\[
\leq b\delta \varepsilon \int_0^T \int_\Omega |D_t^f | \, F|^2 \, dx \, ds + C\left\| \|\nabla \dot{u}\|_{L^{q+1}(0,T;L^{q+1}(\Omega))}^{q+1} \right\|_{L^{q+1}(0,T;L^{q+1}(\Omega))}^2,
\]

where \(C > 0\) depends on \(b, \delta, \varepsilon, q\) and \(|\nabla \zeta|_{L^\infty(W)}\). Note that the first term in the last line can be absorbed by the \(b\delta\) term on the left hand side in \(5.2\) for sufficiently small \(\varepsilon > 0\). The two remaining terms on the right hand side in \(5.2\) can be estimated as follows

\[
k \int_0^t \int_\Omega (\dot{u}^2 + 2\dot{u}) \zeta^2(D_t^f \dot{u})^2 \, dx \, ds + 2k \int_0^t \int_\Omega \zeta^2 \dot{u} \, D_t^f \dot{u} \, dx \, ds
\]

\[
\leq C \left\| \|\dot{u}\|_{L^\infty(0,T;L^\infty(\Omega))} \|\zeta D_t^f \dot{u}\|_{L^2(0,T;L^2(\Omega))}^2 \right\|
\]
due to Sobolev’s embedding theorem we have that \( u \in F \) (for this argument see [46, Chapter 4]). Indeed, since \( \dot{\gamma} u \in L^2(0,T;L^2(\Omega)) \), we can conclude that \( u \in H^1(0,T;H^1(\Omega)) \) and \( \| u \|_{H^1(0,T;H^1(\Omega))} \leq L \), sufficiently small \( \zeta \). Altogether, for sufficiently small \( \varepsilon > 0 \) and \( \bar{m} \), we can achieve that

\[
\| \dot{\gamma} u \|_{L^2(0,T;L^2(\Omega))}^2 + \| \dot{\gamma} (\nabla u) \|_{L^2(0,T;L^2(\Omega))}^2 + \| \ddot{\gamma} (\nabla u) \|_{L^2(0,T;L^2(\Omega))}^2 \leq C (\| \nabla u \|_{L^2(0,T;L^2(\Omega))}^2 + \| \dot{\gamma} u \|_{L^2(0,T;L^2(\Omega))}^2 + \| \ddot{\gamma} u \|_{L^2(0,T;L^2(\Omega))}^2) + \| D^I u \|_{L^2(0,T;L^2(\Omega))}^2 + \| D^I F \|_{L^2(0,T;L^2(\Omega))}^2.
\]

By remembering the definition of \( \zeta \) and Lemma 5.1, we finally arrive at

\[
\| D^I u \|_{L^2(0,T;L^2(\Omega))}^2 + \| D^I F \|_{L^2(0,T;L^2(\Omega))}^2 \leq C (\| \nabla u \|_{L^2(0,T;L^2(\Omega))}^2 + \| D^I u \|_{L^2(0,T;L^2(\Omega))}^2 + \| \ddot{\gamma} u \|_{L^2(0,T;L^2(\Omega))}^2) + \| D^I u \|_{L^2(0,T;L^2(\Omega))}^2 + \| D^I F \|_{L^2(0,T;L^2(\Omega))}^2.
\]

for \( r \in [1, d] \), sufficiently small \( l \) > 0 and sufficiently large \( C > 0 \) which does not depend on \( t \). By employing Lemma 5.1 and Proposition 4.1 we can conclude that \( u \in H^1(0,T;H^2_{\text{loc}}(\Omega)) \) and \( |D^I u|^{\frac{q+1}{q}} (\nabla u) \in L^2(0,T;H^1(\Omega)) \). As a simple consequence of the previous proposition, we can obtain Hölder continuity of \( u \) (for this argument see [46, Chapter 4]). Indeed, since \( F \in L^2(0,T;H^2_{\text{loc}}(\Omega)) \) and \( d \in \{1, 2, 3\} \), due to Sobolev’s embedding theorem we have that \( F \in L^2(0,T;L^6_{\text{loc}}(\Omega)) \). This implies that \( \dot{\gamma} u \in L^{q+1}(0,T;W^{1,3(q+1)}_{\text{loc}}(\Omega)) \). We can then conclude that \( \dot{\gamma} u \in L^{q+1}(0,T;C^{0,\alpha}_{\text{loc}}(\Omega)) \), where \( \alpha = 1 - \frac{1}{q+1} \).

When \( d \in \{1, 2\} \) we can do even better. According to Sobolev’s embedding theorem, \( \dot{\gamma} u \in L^{q+1}(0,T;C^{1,\frac{1}{2}}_{\text{loc}}(\Omega)) \) if \( d = 1 \), and \( \dot{\gamma} u \in L^{q+1}(0,T;C^{0,\gamma}_{\text{loc}}(\Omega)) \) if \( d = 2 \), where \( \gamma \geq 0, 1 \). Altogether, we have

**Corollary 5.3.** Let the assumptions of Theorem 5.2 hold true and additionally \( u_0 \in W^{1,3(q+1)}_{\text{loc}}(\Omega) \). Then

\[
(5.3) \quad u \in \begin{cases} 
W^{1,q+1}(0,T;C^{0,\alpha}_{\text{loc}}(\Omega)) & \text{if } d = 3, \\
W^{1,q+1}(0,T;C^{0,\gamma}_{\text{loc}}(\Omega)) & \text{if } d = 2, \\
W^{1,q+1}(0,T;C^{1,\frac{1}{2}}_{\text{loc}}(\Omega)) & \text{if } d = 1.
\end{cases}
\]

**5.2.1 Neumann problem for the Westervelt equation.** Let us also consider the Neumann problem for the Westervelt equation with strong nonlinear damping:

\[
\begin{cases}
(1 - 2k u) \ddot{u} - c^2 \Delta u - b \, \text{div}(((1 - \delta) + \delta |\nabla \dot{u}|^{q-1}) \nabla \dot{u}) = 2k(\dot{\gamma} u)^2 \quad \text{in } \Omega \times (0,T), \\
c^2 \frac{\partial u}{\partial t} + b((1 - \delta) + \delta |\nabla \dot{u}|^{q-1}) \frac{\partial u}{\partial t} = g \quad \text{on } \partial \Omega \times (0,T), \\
(u, \dot{u})|_{t=0} = (u_0, u_1) \quad \text{on } \overline{\Omega},
\end{cases}
\]

(5.4)
5.3. Interior regularity for the coupled problem

with the same assumptions (4.2) on coefficients. Problem (5.4) is locally well-posed thanks to Theorem 4.10 (see also Remark 4.11).

By inspecting the proof of Proposition 5.2, we immediately obtain a higher interior regularity result for the present model, since the cut-off function used in the proof vanishes near the boundary:

**Corollary 5.4.** Let assumptions (4.2) on coefficients hold true, \( u_0 \in H^2(\Omega) \cap W^{1,q+1}(\Omega), u_1 \in W^{1,q+1}(\Omega) \), and let \( u \) be the weak solution of (5.4). Then \( u \in H^1(0,T; H^2_{\text{loc}}(\Omega)) \) and \( |\nabla \dot{u}|^{q+1} \dot{u} \in L^2(0,T; H^1_{\text{loc}}(\Omega)) \). If additionally \( u_0 \in W^{1,3(q+1)}(\Omega) \), then (5.3) holds.

### 5.3 Interior regularity for the coupled problem

We now return to the acoustic-acoustic coupling problem from Section 4.3. Let us assume that \( \Omega \subset \mathbb{R}^d, d \in \{1,2,3\} \), is a bounded domain with Lipschitz boundary \( \partial \Omega \), and \( \Omega_+ \) a subdomain, representing the lens, such that \( \overline{\Omega}+ \subset \Omega \) and \( \Omega_+ \) has Lipschitz boundary \( \partial \Omega_+ = \Gamma \).

We denote by \( \Omega_- = \Omega \setminus \overline{\Omega}_+ \) the part of the domain representing the fluid region. We then have \( \partial \Omega_- = \Gamma \cup \partial \Omega \).

\( n_+, n_- \) will stand for the unit outer normals to the lens \( \Omega_+ \) and fluid region \( \Omega_- \). Restrictions of a function \( v \) to \( \Omega_{+, -} \) will be denoted by \( v_+, v_- \), and \( \|v\| := v_+ - v_- \) will denote the jump over \( \Gamma \).

Note that the assumption on the regularity of the subdomains will be strengthened to \( C^{1,1} \) when showing higher regularity up to the boundary of the subdomains.

In comparison to assumptions (4.60) from Section 4.3, now the coefficients in the equation will be allowed to jump only over the interface \( \Gamma \):

\[
\begin{align*}
\lambda, k, \varrho, b, \delta &\in L^\infty(\Omega), \\
w_i := w|_{\Omega_i} \in C^1(\Omega_i) &\text{ for } w \in \{b, \varrho, \lambda, \delta, k\}, \ i \in \{+, -\}, \\
\overline{w} := |w|_{L^\infty(\Omega)} &\text{ for } w \in \{b, \varrho, \lambda, \delta, k\}, \ \overline{\delta} < 1, \\
&\exists \varrho, b, \delta : \lambda \geq \Lambda > 0, \ \varrho \geq \underline{\varrho} > 0, \ b \geq \underline{b} > 0, \ \delta \geq \overline{\delta} > 0.
\end{align*}
\]

We assume that \( q \geq 1, q > d - 1 \). Under these assumptions, the strong formulation of (4.61) reads as follows:

\[
\begin{align*}
&\frac{1}{\lambda(x)}(1 - 2k(x)u)\ddot{u} - \text{div}\left(\frac{1}{\varrho(x)} \nabla u\right) - \text{div}(b(x)((1 - \delta(x)) + \delta(x)|\nabla \dot{u}|^{q-1})\nabla \dot{u}) \\
&= \frac{2k(x)}{\lambda(x)}(\dot{u})^2 \text{ in } \Omega_+ \cup \Omega_-, \\
\|u\| &= 0 \text{ on } \Gamma = \partial \Omega_+, \\
\left[\frac{1}{\varrho} \frac{\partial u}{\partial n_+} + b(1 - \delta) \frac{\partial \dot{u}}{\partial n_+} + b\delta|\nabla \dot{u}|^{q-1} \frac{\partial \dot{u}}{\partial n_+}\right] &= 0 \text{ on } \Gamma = \partial \Omega_+, \\
u &= 0 \text{ on } \partial \Omega, \\
(u, \dot{u})|_{t=0} &= (u_0, u_1).
\end{align*}
\]
This problem is locally well-posed thanks to Proposition 4.14.

For simplicity of exposition, higher interior and later boundary regularity will be obtained under the assumption that the coefficients in (5.6) are piecewise constant functions, i.e.

\[
\begin{align*}
\lambda, k, q, b, \delta &\in L^\infty(\Omega), \\
 w_i := w|_{\Omega_i} \text{ is constant, for } w \in \{b, q, \lambda, \delta, k\}, i \in \{+, -\}, \\
 w_i > 0 &\text{ for } w \in \{b, q, \lambda\}, \quad \delta_i \in (0, 1), \quad k_i \in \mathbb{R}.
\end{align*}
\]

We denote \( \omega = \min\{|\omega_+|, |\omega_-|\} \), where \( \omega \in \{b, q, \lambda, \delta, k\} \). Note that now \( \overline{\omega} = |w|_{L^\infty(\Omega)} = \max\{|\omega_+|, |\omega_-|\} \). The proof of Theorem 5.2 can be carried over in a straightforward manner to the coupled problem to show higher interior regularity within each of the subdomains:

**Corollary 5.5.** Assume that \( q \geq 1, q > d-1, u_0|_{\Omega_i} \in H^2(\Omega_i), i \in \{+, -\}, u_0, u_1 \in W^{1,q+1}_0(\Omega) \) and assumptions (5.7) on the coefficients hold true. Let \( u \) be the weak solution of (5.6). Then \( u_i \in H^1(0,T;H^2_{loc}(\Omega_i)) \) and \( |\nabla u_i|^{q-1} \nabla u_i \in L^2(0,T;H^1_{loc}(\Omega_i)), i \in \{+, -\} \). If additionally \( u_0|_{\Omega_i} \in W^{1,3(q+1)}(\Omega_i) \), then (5.3) holds with \( u \) replaced by \( u_i \) and \( \Omega \) by \( \Omega_i, i \in \{+, -\} \).

### 5.3.1 Neumann problem for the coupled system.

Let us also consider the coupled problem with Neumann boundary conditions on the outer boundary of the fluid subdomain:

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{1}{\lambda(x)}(1 - 2k(x)u)\ddot{u} - \text{div}\left(\frac{1}{\varrho(x)}\nabla u\right) - \text{div}(b(x)((1 - \delta(x)) + \delta(x)|\nabla \dot{u}|^{q-1})\nabla \dot{u}) \\
\frac{1}{\varrho(x)}(\dot{u})^2 \quad \text{in } \Omega_+ \cup \Omega_-,
\end{array} \right.
\]

\[
\left\{ \begin{array}{l}
\|u\| = 0 \text{ on } \Gamma = \partial \Omega_+,
\end{array} \right.
\]

\[
\left\{ \begin{array}{l}
\frac{1}{\varrho(x)}\frac{\partial u}{\partial n_+} + b(1 - \delta)\frac{\partial \dot{u}}{\partial n_+} + b\delta|\nabla \dot{u}|^{q-1}\frac{\partial \dot{u}}{\partial n_+} = 0 \text{ on } \Gamma = \partial \Omega_+,
\end{array} \right.
\]

\[
\left\{ \begin{array}{l}
\dot{u}, \tilde{u} \mid_{t=0} = (u_0, u_1).
\end{array} \right.
\]

Recall that the well-posedness result from Corollary 4.16 holds. The interior regularity result can again be transferred from Corollary 5.3 to the present model:

**Corollary 5.6.** Let \( q > d-1, q \geq 1, \) and assumptions (5.7) on the coefficients hold true, let \( u_0|_{\Omega_i} \in H^2(\Omega_i), i \in \{+, -\}, u_0, u_1 \in W^{1,q+1}_0(\Omega) \), and let \( u \) be the weak solution of (5.8). Then \( u_i \in H^1(0,T;H^2_{loc}(\Omega_i)) \) and \( |\nabla u_i|^{q-1} \nabla u_i \in L^2(0,T;H^1_{loc}(\Omega_i)), i \in \{+, -\} \). If additionally \( u_0|_{\Omega_i} \in W^{1,3(q+1)}(\Omega_i) \), then (5.3) holds with \( u \) replaced by \( u_i \) and \( \Omega \) by \( \Omega_i, i \in \{+, -\} \).

### 5.4 Boundary regularity for the coupled problem.

We will show next that the \( H^2 \)-regularity result can be extended up to the boundary of each of the subdomains under the assumption that the gradient of \( u \) is essentially bounded in time and space on the whole domain. For this property to hold, we will need to smoothen out boundaries of the subdomains, i.e. assume that they are \( C^{1,1} \) regular. The proof will expand on the approach taken in [3] Lemma 3.6.]
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**Theorem 5.7.** (Boundary $H^2$-regularity) Let the assumptions of Corollary 5.5 hold and let $\partial \Omega$ and $\Gamma = \partial \Omega_+$ be $C^{1,1}$ regular. If $u \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega))$ and $\|\nabla u\|_{L^\infty(0,T;L^\infty(\Omega))}$ is sufficiently small, then

$$u_i \in H^1(0,T;H^2(\Omega_i)), \ i \in \{+, -\}.$$  

**Proof.** We will only show that $u_+ \in H^1(0,T;H^2(\Omega_+))$, since $u_- \in H^1(0,T;H^2(\Omega_-))$ follows analogously.

**Step 1: Straightening the boundary.** We begin by straightening the boundary through the change of coordinates near a boundary point (cf. [20, Theorem 4, Section 6.3.2]). Choose any point $x_0 \in \partial \Omega_+$. There exists a ball $B = B_r(x_0)$ for some $r > 0$ and a $C^{1,1}$-diffeomorphism $\Psi : B \to \Psi(B) \subset \mathbb{R}^d$ such that $\det|\nabla \Psi| = 1$, $U' = \Psi(B)$ is an open set, $\Psi(B \cap \Omega_+) \subset \mathbb{R}^d_+$ and $\Psi(B \cap \Gamma) \subset \partial \mathbb{R}^d_+$, where $\mathbb{R}^d_+$ is the half-space in the new coordinates.

We change the variables and write

$$y = \Psi(x), \ x \in B,$$

$$x = \Phi(y), \ y \in U'.$$

Then we have $\Psi(B \cap \Omega_+) = \{y \in U' : y_n > 0\}$. We denote $B^+ = B_2 \cap \Omega_+, \ G = \Psi(B_2(x_0)), \ G^+ = \Psi(B^+)$. Then $G \subset U'$ and $G^+ \subset G$. We define

$$w(y,t) := u(\Phi(y), t), \ (y,t) \in U' \times [0,T].$$

It immediately follows that $w(t) := w(., t) \in W^{1,q+1}(U')$. We now transform the original equation on $B \times [0,T]$ into an equation on $U' \times [0,T]$:

$$
\int_{U'} \left\{ \frac{1}{\lambda} \left( 1 - 2k \omega(t) \right) \hat{\omega}(t) \phi + \sum_{i,j=1}^d \left( \hat{\sigma}_{ij} D_i w(t) D_j \phi + \hat{\xi}_{ij} D_i \hat{\omega}(t) D_j \phi \right) \\
+ \left| J^\Phi \nabla \omega(t) \right|^{-1} \hat{\eta}_{ij} D_i \hat{\omega}(t) D_j \phi \right\} dy = 0,
$$

for a.e. $t \in [0,T]$, and all $\phi \in W^{1,q+1}_0(U')$, where $D_i w = \frac{\partial w}{\partial y_i}$, and

$$\hat{\lambda}(y) = \lambda(\Phi(y)), \ \hat{k}(y) = k(\Phi(y)),
$$

$$\hat{\sigma}_{ij} = \sum_{r=1}^d \frac{1}{\varrho(\Phi(y))} \frac{\partial \Phi_i}{\partial x_r}(\Phi(y)) \frac{\partial \Phi_j}{\partial x_r}(\Phi(y)).$$

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Chapter 5. Higher regularity

(5.10) \[
\hat{\xi}_{ij}(y) = \sum_{r=1}^{d} b(\Phi(y))(1 - \delta(\Phi(y))) \frac{\partial \Phi_i}{\partial x_r}(\Phi(y)) \frac{\partial \Phi_j}{\partial x_r}(\Phi(y)), \\
\hat{\eta}_{ij}(y) = \sum_{r=1}^{d} b(\Phi(y))\delta(\Phi(y)) \frac{\partial \Phi_i}{\partial x_r}(\Phi(y)) \frac{\partial \Phi_j}{\partial x_r}(\Phi(y)).
\]

Note that \(D_r \hat{\sigma}_{ij}, D_r \hat{\xi}_{ij}, D_r \hat{\eta}_{ij} \in L^\infty(U')\) for \(r \in \{1, \ldots, d-1\}\) since \(\Psi\) and \(\Phi\) are \(C^{1,1}\) mappings which means they are \(W^{2,\infty}\)-regular (see Theorem 3.2). It can be shown (for details see [20 Section 6.3.2]) that

(5.11) \[
\sum_{i,j=1}^{d} \hat{\sigma}_{ij} \phi_i \phi_j \geq K_1|\phi|^2, \quad \sum_{i,j=1}^{d} \hat{\xi}_{ij} \phi_i \phi_j \geq K_1|\phi|^2, \\
\sum_{i,j=1}^{d} \hat{\eta}_{ij} \phi_i \phi_j \geq K_1|\phi|^2, \quad \forall (y, \phi) \in U' \times \mathbb{R}^d.
\]

Next, we choose a domain \(W'\) such that \(G \subset \subset W' \subset \subset U'\) and select a cut-off function such that

\[
\begin{cases}
\zeta = 1 \text{ on } G, & \zeta = 0 \text{ on } \mathbb{R}^d \setminus W', \\
0 \leq \zeta \leq 1.
\end{cases}
\]

Let \(|l| > 0\) be small and choose \(r \in \{1, \ldots, d-1\}\). Note that since we consider directions parallel to the interface now, we have \(D_r^l \lambda = 0, D_r^l \hat{k} = 0\) and that there exists a constant \(K_2 > 0\) such that

(5.12) \[
|D_r^l \hat{\sigma}_{ij}(y)| < K_2, \quad |D_r^l \hat{\xi}_{ij}(y)| < K_2, \quad |D_r^l \hat{\eta}_{ij}(y)| < K_2
\]

for a.e. \(y \in W', 1 \leq i, j \leq d\) and sufficiently small \(|l|\).

Step 2: Existence of second order derivatives \(D_j D_l w \in H^1(0, T; L^2(G^+)), j \neq d\). We then use \(\phi = -D_r^{-1}(\xi D_l^i \hat{\omega}(t))\) as a test function in (5.9), which, after integration with respect to time, results in

\[
\begin{align*}
\frac{1}{2} \int_{W'} \left( 1 - 2 \hat{k}^i w' \right) (\xi D_l^i \hat{\omega}) dy + \frac{1}{2} \int_{W'} \sum_{i,j=1}^{d} \hat{\sigma}_{ij} \xi^2 D_l^i D_l^j \hat{\omega} dy \\
+ \int_{0}^{t} \int_{W'} \sum_{i,j=1}^{d} \hat{\xi}_{ij} \xi^2 D_l^i D_l^j \hat{\omega} dy ds \\
= \int_{0}^{t} \int_{W'} \frac{2 \hat{k}^i}{\hat{l}^i} D_l^i \hat{\omega} \xi^2 D_l^j \hat{\omega} dy ds + \int_{0}^{t} \int_{W'} \frac{\hat{k}^i}{\hat{l}^i} (\hat{\omega}^2 + 2 \hat{\omega}) (\xi D_l^i \hat{\omega})^2 dy ds \\
- \int_{0}^{t} \int_{W'} \sum_{i,j=1}^{d} D_l^j (\hat{\sigma}_{ij}) D_l^j \hat{\omega} \hat{\xi} (\xi^2 D_l^i \hat{\omega} + 2 \xi D_l^i \hat{\xi} D_l^j \hat{\omega}) dy ds \\
- 2 \int_{0}^{t} \int_{W'} \sum_{i,j=1}^{d} \hat{\sigma}_{ij} D_l^i D_l^i \hat{\omega} \xi D_l^j \hat{\omega} dy ds
\end{align*}
\]
5.4. Boundary regularity for the coupled problem

\[-\int_0^t \int_{\Omega'} \sum_{i,j=1}^d D^t_i(\dot{\xi}_{ij}) D_i \dot{w} (\zeta^2 D^t_j D_j \dot{w} + 2\zeta D_j \zeta D^t_j \dot{w}) \, dy \, ds\]

\[-2 \int_0^t \int_{\Omega'} \sum_{i,j=1}^d \dot{\xi}_{ij} D_i \dot{w} \zeta D_j \zeta D^t_i \dot{w} \, dy \, ds\]

\[-\int_0^t \int_{\Omega'} \sum_{i,j=1}^d D^t_i(|J^T_\phi \nabla \dot{w}|^{q-1} \dot{\eta}_{ij} D_i \dot{w}) (\zeta^2 D^t_j D_j \dot{w} + 2\zeta D_j \zeta D^t_j \dot{w}) \, dy \, ds.\]

We can estimate the last term on the right hand side as follows

\[-\int_0^t \int_{\Omega'} \sum_{i,j=1}^d D^t_i(|J^T_\phi \nabla \dot{w}|^{q-1} \dot{\eta}_{ij} D_i \dot{w}) (\zeta^2 D^t_j D_j \dot{w} + 2\zeta D_j \zeta D^t_j \dot{w}) \, dy \, ds\]

\[= - \int_0^t \int_{\Omega'} \sum_{i,j=1}^d |(J^T_\phi \nabla \dot{w})|^{q-1} \dot{\eta}_{ij} D_i \dot{w} (\zeta^2 D^t_j D_j \dot{w} + 2\zeta (D_j \zeta D^t_j \dot{w}) \, dy \, ds\]

\[-\int_0^t \int_{\Omega'} \sum_{i,j=1}^d D^t_i(|J^T_\phi \nabla \dot{w}|^{q-1} \dot{\eta}_{ij}) D_i \dot{w} (\zeta^2 D^t_j D_j \dot{w} + 2\zeta (D_j \zeta D^t_j \dot{w}) \, dy \, ds\]

\[\leq C \|\nabla \dot{w}\|^q_{L^\infty(0,T; L^\infty(W'))} \left( \sum_{i=1}^d \|\zeta D^t_i D_i \dot{w}\|_{L^2(0,T; L^2(W'))}^2 + \|\zeta \dot{w}\|_{L^\infty(0,T; L^\infty(W'))} \|\nabla \dot{w}\|^2_{L^2(0,T; L^2(W'))} \right)
\]

with \(C\) independent of \(\zeta\). Due to (5.12) the rest of the terms on the right hand side can be estimated analogously to the estimates in the proof of Theorem 5.2 which for sufficiently small \(\|\nabla \dot{w}\|_{L^\infty(0,T; L^\infty(W'))}\) leads to

\[\|\zeta D^t_i \dot{w}\|_{L^\infty(0,T; L^\infty(W'))}^2 + \sum_{i=1}^d \|\zeta D^t_i D_i \dot{w}\|_{L^\infty(0,T; L^2(W'))}^2 + \sum_{i=1}^d \|\zeta D^t_i D_i \dot{w}\|_{L^2(0,T; L^2(W'))}^2 \leq C((1 + \|\dot{w}\|_{L^\infty(0,T; L^\infty(W'))}) \|\nabla \dot{w}\|^2_{L^\infty(0,T; L^2(W'))} + (1 + \|\nabla \dot{w}\|_{L^\infty(0,T; L^\infty(W'))}) \|\nabla \dot{w}\|^2_{L^2(0,T; L^2(W'))} + |\dot{w}(0)|^2_{L^2(W')} + |w(0)|^2_{H^2(W')}\).

Recalling the definition of \(\zeta\) and employing Lemma 5.1 yields \(D_j D_i \dot{w} \in H^1(0,T; L^2(G^+))\) for \(1 \leq i \leq d, 1 \leq j \leq d - 1\).

**Step 3: Existence of second order derivative** \(D_d D_d \dot{w} \in H^1(0,T; L^2(G^+))\). It remains to show that \(D_{dd} \dot{w} := D_d D_d \dot{w} \in H^1(0,T; L^2(G^+))\). From (5.9), after integration by parts, we obtain

\[
\int_{G^+} \left\{ \tilde{\sigma}_{dd} D_d \dot{w}(t) + \tilde{\xi}_{dd} D_d \dot{w}(t) + |J^T_\phi \nabla \dot{w}(t)|^{q-1} \tilde{\eta}_{dd} D_d \dot{w}(t) \right\} D_d \phi \, dy
\]

\[= \int_{G^+} \left\{ \frac{1}{\lambda} \left(1 - 2\tilde{k} w(t))\dot{w}(t) + \frac{2\tilde{k}}{\lambda} \dot{w}(t) \right)^2
\]

\[+ \sum_{j=1}^{d-1} \sum_{i=1}^d \left(D_j(\tilde{\sigma}_{ij} D_i \dot{w}(t)) + D_j(\tilde{\xi}_{ij} D_i \dot{w}(t)) + D_j(|J^T_\phi \nabla \dot{w}(t)|^{q-1} \tilde{\eta}_{ij} D_i \dot{w}(t)) \right) \right\} \phi \, dy,
\]

\[= \int_{G^+} \tilde{f}(w)(t) \phi \, dy,
\]
for \( \phi \in C_0^\infty(G^+) \), a.e. in \([0, T]\). Since the right hand side of the equation is well-defined, we conclude that for a.e. \( t \in [0, T] \) the weak derivative of
\[
\dot{\sigma}_{dd} D_d w(t) + \dot{\xi}_{dd} D_d \dot{w}(t) + |J_\phi^T \nabla \dot{w}|^{q-1} \eta_{dd} D_d \dot{w}(t)
\]
with respect to \( y_d \) exists on \( G^+ \). Furthermore, for a.e. \( t \in [0, T] \) the weak derivative satisfies
\[
(5.14) \quad - D_d(\dot{\sigma}_{dd} D_d w(t) + \dot{\xi}_{dd} D_d \dot{w}(t) + |J_\phi^T \nabla \dot{w}|^{q-1} \eta_{dd} D_d \dot{w}(t)) = \dot{f}(w)(t)
\]
on \( G^+ \). From what we have shown, it follows that \( \dot{f}(w) \in L^2(0, T; L^2(G^+)) \). We set
\[
z(t) := \dot{\sigma}_{dd} D_d w(t) + \dot{\xi}_{dd} D_d \dot{w}(t) + |J_\phi^T \nabla \dot{w}|^{q-1} \eta_{dd} D_d \dot{w}(t),
\]
and
\[
\dot{\xi}_{dd}(t, D_d \dot{w}(t)) := \dot{\xi}_{dd} + |J_\phi^T \nabla \dot{w}(t)|^{q-1} \eta_{dd}.
\]
(suppressing in the notation dependence on \( D_d \dot{w}(t), \ldots, D_{d-1} \dot{w}(t) \) which we already know to be smooth anyway) so that relation \((5.14)\) reads
\[
-D_d z(t) = \dot{f}(w)(t),
\]
where
\[
z(t) = \dot{\sigma}_{dd} D_d w(t) + \dot{\xi}_{dd}(t, D_d \dot{w}(t)) D_d \dot{w}(t).
\]
Since \( \dot{f}(w) \in L^2(0, T; L^2(G^+)) \), and using the fact that \( D_j D_i w \in H^1(0, T; L^2(G^+)) \) for \( 1 \leq i \leq d, 1 \leq j \leq d - 1 \), we have \( z \in L^2(0, T; H^1(G^+)) \). On the other hand, due to assuming that \( u \in W^{1,\infty}(0, T; W^{1,\infty}(\Omega)) \), we know also that \( z \in L^\infty(0, T; L^\infty(G^+)) \). Therefore, we conclude that
\[
z \in L^2(0, T; H^1(G^+)) \cap L^\infty(0, T; L^\infty(G^+)).
\]
Since \((5.15)\) represents an ODE (pointwise a.e. in space) for \( D_d w(t) \), it can be resolved as follows:
\[
D_d w(t)
\]
\[
= \exp \left(- \int_0^t \dot{\sigma}_{dd} d\tau \right) \left( \int_0^t \frac{z(\tau)}{\xi_{dd}(\tau, D_d \dot{w}(\tau))} \exp \left(- \int_0^{\tau} \frac{\dot{\sigma}_{dd}}{\xi_{dd}(\rho, D_d \dot{w}(\rho))} d\rho \right) d\tau + D_d w(0) \right),
\]
and then also \( D_d \dot{w}(t) \) can be expressed in terms of \( z, \dot{\sigma}_{dd}, \dot{\xi}_{dd} \):
\[
D_d \dot{w}(t) = \frac{z(t)}{\xi_{dd}(t, D_d \dot{w}(t))} - \frac{\dot{\sigma}_{dd}}{\xi_{dd}(t, D_d \dot{w}(t))} \exp \left(- \int_0^t \frac{\dot{\sigma}_{dd}}{\xi_{dd}(\tau, D_d \dot{w}(\tau))} d\tau \right) \left( \int_0^t \frac{z(\tau)}{\xi_{dd}(\tau, D_d \dot{w}(\tau))} \exp \left(- \int_0^{\tau} \frac{\dot{\sigma}_{dd}}{\xi_{dd}(\rho, D_d \dot{w}(\rho))} d\rho \right) d\tau + D_d w(0) \right).
\]
Since the right hand side depends on \( D_d \dot{w}(t) \), this is a fixed point equation rather than an explicit expression for \( D_d \dot{w}(t) \). We therefore consider the fixed point operator \( T : M \to M \), defined by the right hand side of \((5.17)\), i.e.,
\[
T = S \circ R,
\]
where \( S \) is the operator that sends \( w \) to \( \dot{f}(w) \) and \( R \) is the operator that sends \( w \) to \( D_d \dot{w}(t) \) for a.e. \( t \in [0, T] \). The fixed point theorem then guarantees the existence of a solution to our initial value problem.
where $\mathcal{R} : M \to \tilde{M}$, $\tilde{M} = \{ \phi \in M : \phi \geq K_1 \}$, is the superposition operator associated with $\xi_{dd}$, i.e., $\mathcal{R}(v)(t) = \xi_{dd}(t, v(t))$, and $S : \tilde{M} \to M$

$$S(r)(t) = \frac{z(t)}{r(t)} - \frac{\hat{\sigma}_{dd}}{r(t)} \exp \left( - \int_0^t \frac{\hat{\sigma}_{dd}}{r(\tau)} \, d\tau \right) \left( \int_0^t \frac{z(\tau)}{r(\tau)} \exp \left( \int_0^\tau \frac{\hat{\sigma}_{dd}}{r(\rho)} \, d\rho \right) \, d\tau + D_d w(0) \right).$$

Note that both $\hat{\sigma}_{dd}$ and $\xi_{dd}$ are bounded from below by $K_1$ due to (5.11). We can then conclude that $\mathcal{T}$ is a self-mapping on

$$M = M_0 = L^\infty(0, T; L^\infty(G^+)),$$

and on

$$M = M_1 = L^2(0, T; H^1(G^+)) \cap L^\infty(0, T; L^\infty(G^+)),$$

since $z \in L^2(0, T; H^1(G^+)) \cap L^\infty(0, T; L^\infty(G^+))$. Moreover,

$$\mathcal{R}'(D_d \hat{w}) = \frac{\partial}{\partial v} \left( \xi_{dd} + |J_q^\gamma(D_1 \hat{w}, \ldots, D_{d-1} \hat{w}, v)|^{-1} \hat{\eta}_{dd} \right) |_{v = D_d \hat{w}}$$

is small if $\|\nabla u_0\|_{L^\infty(0, T; L^\infty(\Omega))}$ and therefore $\|\nabla \hat{w}\|_{L^\infty(0, T; L^\infty(G^+))}$ is small. This implies that $\mathcal{T}$ is a contraction on

$$\tilde{M}_0 = \{ v \in M_0 : \| v - D_d \hat{w}\|_{L^\infty(0, T; L^\infty(G^+))} \leq \gamma \}$$

and on

$$\tilde{M}_1 = \{ v \in M_1 : \| v - D_d \hat{w}\|_{L^\infty(0, T; L^\infty(G^+))} \leq \gamma \}.$$

for $\gamma$ sufficiently small. Thus, the fixed point equation $v = \mathcal{T}(v)$ has a unique solution $v_0$ in $\tilde{M}_0$ and it also has a unique solution $v_1$ in $\tilde{M}_1$, and both have to coincide $v_1 = v_0$ by uniqueness on $\tilde{M}_0 \supseteq \tilde{M}_1$. On the other hand, obviously $D_d \hat{w}$ lies in $\tilde{M}_0$, solves this fixed point equation and thus has to coincide with $v_0$, hence also with $v_1$. This proves that $D_d \hat{w} \in M_1 \subseteq L^2(0, T; H^1(G^+)).$

By transforming $w$ back to $u$, we can conclude that $u \in H^1(0, T; H^2(B^+))$. The assertion then follows from the fact that the boundary is compact and can be covered by a finite set of balls $\{ B_{r_i/2}(x_i) \}_{i=1}^N$.

Higher boundary regularity was obtained under the assumption that $u$ belongs to the space $W^{1,\infty}(0, T; W^{1,\infty}(\Omega))$; this was necessitated by the presence of the $q$-Laplace damping term in the equation. The assumption is equivalent to assuming Lipschitz continuity of $u$ in time and space, i.e. $u \in C^{0,1}(0, T; C^{0,1}(\Omega))$ (see Theorem 3.2). Recall that, thanks to Theorem 4.14 and the Sobolev embedding $W^{1,q+1}(\Omega) \hookrightarrow C^{0,1+\frac{q}{q+1}}(\Omega)$, we know that $u$ is Hölder continuous in space, i.e. $u \in C^{0,1}(0, T; C^{0,1-\frac{q}{q+1}}(\Omega))$.

### 5.4.1 Neumann problem for the coupled system.

It remains to show $H^2$-regularity up to the boundary for the Neumann problem (5.8).

**Theorem 5.8.** Let the assumptions of Corollary 5.6 hold, let $\partial \Omega$ and $\Gamma = \partial \Omega_+$ be $C^{1,1}$ regular and let $u$ be the weak solution of (5.8). Furthermore, assume that $g \in L^2(0, T; H^{1/2}(\partial \Omega))$. If $u \in W^{1,\infty}(0, T; W^{1,\infty}(\Omega))$ and $\|\nabla \hat{u}\|_{L^\infty(0, T; L^\infty(\Omega))}$ is sufficiently small, then

$$u_i \in H^1(0, T; H^2(\Omega_i)), \ i \in \{+,-,\}.$$
Proof. We will show that \( u_- \in H^1(0,T;H^2(\Omega_-)) \), since the regularity on \( \Omega_+ \) follows as in the proof of Theorem 5.7. We begin again as before, by straightening the boundary around \( x_0 \in \partial \Omega_- \setminus \partial \Omega_+ \). There exists a ball \( B = B_r(x_0) \) for some \( r > 0 \) and a \( C^{1,1} \)-diffeomorphism \( \Psi : B \to \Psi(B) \subset \mathbb{R}^d \) such that \( \det \nabla \Psi = 1 \), \( U' = \Psi(\Omega_-) \subset \mathbb{R}^d \) is an open set, and \( \Psi(B \cap \partial \Omega_-) \subset \partial \mathbb{R}^d_+ \). We change the variables and write

\[
y = \Psi(x), \quad x \in B,
x = \Phi(y), \quad y \in U'.
\]

We denote \( G := \Psi(B_r \cap \Omega_-) \subset \subset U' \). We define

\[
w(y,t) := u(\Phi(y),t), \quad (y,t) \in U' \times [0,T],
\]

and transform the orginal equation from \( (B \cap \Omega) \times [0,T] \) to \( U' \times [0,T] \):

\[
\int_{U'} \left\{ \frac{1}{\lambda} (1 - 2\hat{k}w(t))\hat{w}(t)\phi + \sum_{i,j=1}^d \left( \hat{\sigma}_{ij} D_i w(t) D_j \phi + \hat{\xi}_{ij} D_i \hat{w}(t) D_j \phi \right) + |J^T_k \nabla \hat{w}(t)|^{q-1} \hat{\eta}_{ij} D_i \hat{w}(t) D_j \phi \right\} dy = \int_{\partial U'} g \phi |J^T_k n|_{\mathbb{R}^d} dx,
\]

for a.e. \( t \in [0,T] \), and all \( \phi \in W^{1,q+1}(U') \), and \( \hat{\lambda}, \hat{k}, \hat{\sigma}_{ij}, \hat{\xi}_{ij}, \text{ and } \hat{\eta}_{ij} \) defined as in (5.10).

We then again use \( \phi = -D_{r^{-1}}(\zeta^2 D^l_\tau \hat{w}(t)) \), \( r \in \{1, \ldots, d-1\} \) as a test function and proceed with the estimates like in the proof of Theorem 5.7. The only difference here is the need to estimate the boundary integral over \( \partial U' \) appearing in the weak form, therefore we focus our attention solely on estimating this term:

\[
- \int_0^t \int_{\partial U'} g D_{r^{-1}}(\zeta^2 D^l_\tau \hat{u}) |J^T_k n| \, dx \, ds
= \int_0^t \int_{B \cap \partial \Omega_-} D_{r^{-1}}(\zeta^2 D^l_\tau \hat{u}) \, dx \, ds
\leq C \|g\|_{L^2(0,T;H^{1/2}(\partial \Omega_-))} \|\zeta^2 D^l_\tau \hat{u}\|_{L^2(0,T;H^{-1/2}(\partial \Omega_-))}
\leq C \|g\|_{L^2(0,T;H^{1/2}(\partial \Omega_-))} \|\zeta^2 D^l_\tau \hat{u}\|_{L^2(0,T;H^{1}(\Omega_-))}
\leq \varepsilon \|\zeta^2 D^l_\tau \hat{u}\|_{L^2(0,T;H^{-1}(\Omega_-))},
\]

for some conveniently chosen \( \varepsilon > 0 \). Showing that \( D_{\hat{\phi}w} \in H^1(0,T;L^2(G)) \) follows as in the proof of Theorem 5.7, since we use \( \phi \in C^0_\infty(G) \) in (5.18). \( \square \)
CHAPTER 6

Sensitivity analysis for shape optimization of a focusing acoustic lens in lithotripsy

The present chapter is concerned with a shape optimization problem arising in lithotripsy, where an optimal focusing of the ultrasound waves is needed in order to concentrate the ultrasound pressure on the kidney stone and avoid lesions of the surrounding tissue (see fig. 1.1). In particular, we are interested in obtaining the first order sensitivity within this context, which will serve as a basis for future development of an efficient optimization algorithm. For some first steps taken in the direction of mathematically based optimization of the focusing lens, we refer to the thesis [61] (see also [38]).

The approach to computing the shape derivative that we will take follows the general framework developed by Ito, Kunisch, and Peichl in [30]. For its extension to a time dependent setting, we refer to [37]. The main advantage of this variational approach is that it allows to compute the shape derivative of a cost functional without the need to employ the shape derivative of the state. However, as we will see, assumptions of certain spatial regularity of the primal and the adjoint state will be required to obtain the derivative, in particular for obtaining its strong form in terms of boundary integrals.

6.1 Problem formulation

Let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, be a fixed bounded domain with Lipschitz boundary $\partial \Omega$, and, as in the previous chapter, $\Omega_+$ a subdomain, representing the lens, such that $\Omega_+ \subset \Omega$ and $\Omega_+$ has Lipschitz boundary $\partial \Omega_+ = \Gamma$. We denote by $\Omega_- = \Omega \setminus \Omega_+$ the part of the domain representing the fluid region (see fig. 5.2). We then have $\partial \Omega_- = \Gamma \cup \partial \Omega$. Note that the assumptions on the regularity of the subdomains will eventually have to be strengthened to $C^{1,1}$ in order to express the shape derivative in terms of the boundary integrals over the interface $\Gamma$.

We consider the following optimization problem

$$
\begin{align*}
\min_{\Omega_+ \in O_{ad}} \min_{u \in L^2(\Omega \times [0,T])} J(u, \Omega_+) & \equiv \min_{\Omega_+ \in O_{ad}} \int_0^T \int_\Omega (u - u_d)^2 \, dx \, ds,
\end{align*}
$$

subject to the constraint

$$
\begin{align*}
\int_0^T \int_\Omega \left\{ \frac{1}{\alpha(x)}(1 - 2k(x)u)\dot{u}\phi + \frac{1}{\alpha(x)}\nabla u \cdot \nabla \phi + b(x)(1 - \delta(x))\nabla \dot{u} \cdot \nabla \phi 
+ b(x)\delta(x)|\nabla \dot{u}|^{q-1} \nabla \dot{u} \cdot \nabla \phi - \frac{2k(x)}{\alpha(x)}(\dot{u})^2 \phi \right\} \, dx \, ds = 0
\end{align*}
$$

holds for all test functions $\phi \in L^2(0,T;W^{1,q+1}_0(\Omega))$. 

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with \((u, \dot{u})|_{t=0} = (u_0, u_1)\), where
\[
(u_0, u_1) \in \left\{ \left( u_0, u_1 \right) \in W_0^{1,q+1}(\Omega) \times W_0^{1,q+1}(\Omega) :
\begin{align*}
|u_1|^2_{L^2(\Omega)} + |\nabla u_0|^2_{L^2(\Omega)} + |\nabla u_1|^2_{L^{q+1}(\Omega)} + |\nabla u_0|^2_{L^{q+1}(\Omega)} & \leq \kappa_T^2
\end{align*}
\right\}.
\]
final time \(T\) and the size of the initial data \(\kappa_T\) are determined according to Proposition (4.14).
Dependence on \(\Omega_+\) is not directly visible in (6.1), however this dependence is present via the PDE (i.e. via the equality constraint) as will be explained below.

The initial-boundary value problem (4.61) was already stated in Chapter 4, however we have recalled it here for easier readability.

To formulate the problem it is sufficient to assume that coefficients \(\lambda, k, \varrho, b, \delta\) satisfy (5.5), we restate the assumptions here for convenience of the reader:
\[
\begin{align*}
\lambda, k, \varrho, b, \delta & \in L^\infty(\Omega), \\
w_i & := w_i|_{\Omega_+} \in C^1(\Omega_i) \text{ for } w \in \{b, \varrho, \lambda, \delta, k\}, i \in \{+, -\}, \\
\overline{w} & := |w|_{L^\infty(\Omega)} \text{ for } w \in \{b, \varrho, \lambda, \delta, k\}, \ \overline{\delta} < 1, \\
\exists \varrho, b, \overline{\delta}, \delta & : \lambda \geq \overline{\Lambda} > 0, \ \varrho \geq \rho > 0, \ b \geq \overline{b} > 0, \ \delta \geq \overline{\delta} > 0.
\end{align*}
\]
We will, however, eventually return to piecewise constant coefficients (5.7) when calculating the shape derivative. As a matter of fact, dependence on \(\Omega_+\) within the minimization problem (6.1) is just via this interface \(\Gamma = \partial \Omega_+\) where discontinuity of the coefficients is allowed to take place.

In (6.1), \(u_d \in L^2(0,T; L^2(\Omega))\) denotes the desired acoustic pressure and \(\mathcal{O}_{ad}\) represents the set of admissible domains and is defined as follows:
\[
\mathcal{O}_{ad} = \{ \Omega_+ : \Omega_+ \subset \Omega, \ \Omega_+ \text{ is open and Lipschitz with uniform Lipschitz constant } L_\mathcal{O} \}.
\]
We assume that \(q \geq 1, q > d - 1\) for the state and the adjoint problem to be well-posed; however we will need to strengthen this assumption later to \(q > 2\) in order to prove certain properties needed for the characterization of the shape derivative.

Note that, due to assumptions (5.5) on the coefficients, (4.61) can be rewritten as
\[
\begin{align*}
\int_0^T \int_\Omega \{ \lambda(x)(1 - 2k(x)u)\dot{u} + \varrho(x)\nabla u \cdot \nabla \phi + b(x)(1 - \delta(x))\nabla \dot{u} \cdot \nabla \phi \\
+ b(x)\delta(x)|\nabla \dot{u}|^{q-1}\nabla \dot{u} \cdot \nabla \phi - 2k(x)\lambda(x)(\dot{u})^2\phi \} \, dx \, ds = 0
\end{align*}
\]
holds for all test functions \(\phi \in L^2(0,T; W_0^{1,q+1}(\Omega))\),
with \((u, \dot{u})|_{t=0} = (u_0, u_1)\), where \(\dot{w}(x) = \chi_{\Omega_+} \frac{\dot{u}}{w_+} + (1 - \chi_{\Omega_+}) \frac{\dot{u}}{w_-}\) for \(w \in \{\lambda, \varrho\}\) and \(\dot{w}(x) = \chi_{\Omega_+} w_+ + (1 - \chi_{\Omega_+}) w_-\) for \(w \in \{k, \varrho, \delta\}\).

Let us also restate the strong form of the PDE constraint:
\[
\begin{align*}
\frac{1}{\lambda(x)}(1 - 2k(x)u)\ddot{u} - \text{div}\left( \frac{1}{\varrho(x)} \nabla u \right) - \text{div}(b(x)((1 - \delta(x)) + \delta(x)|\nabla \dot{u}|^{q-1})\nabla \dot{u}) \\
= 2k(x)\lambda(x)(\dot{u})^2 \text{ in } \Omega_+ \cup \Omega_-,
\end{align*}
\]
\[
\begin{align*}
\|u\| & = 0 \text{ on } \Gamma = \partial \Omega_+,
\left[ \frac{1}{\varrho(x)} \frac{\partial u}{\partial n_+} + b(x)(1 - \delta(x)) \frac{\partial u}{\partial n_+} + b(x)\delta(x)|\nabla \dot{u}|^{q-1} \frac{\partial u}{\partial n_+} \right] & = 0 \text{ on } \Gamma = \partial \Omega_+,
\end{align*}
\]
\[
\begin{align*}
u & = 0 \text{ on } \partial \Omega,
\end{align*}
\]
\[
\begin{align*}
(u, \dot{u})|_{t=0} = (u_0, u_1).
\end{align*}
\]
6.1. Problem formulation

Note that the equivalence of formulations (4.61) and (6.2) for sufficiently regular solutions is due to the fact that (see, for example, [5, 9]):

\[ H^1(\Omega) = \{ v \in L^2(\Omega) : v|_{\Omega_i} = v_i \in H^1(\Omega_i), \; i \in \{+, -\}, \; \text{tr}_{\Gamma} v_+ = \text{tr}_{\Gamma} v_- \}. \]

For simplicity of exposition we consider homogeneous Dirichlet boundary conditions on the outer boundary \( \partial \Omega \). We mention in passing that Neumann and absorbing boundary conditions as in [37, 51] and Chapter 4 could be easily incorporated here as well, with some additional terms involving the Neumann boundary excitation in the analysis of the state equation, but with no changes in the shape derivative itself, since the outer boundary will not be subject to modifications.

Some difficulties related to the model (5.6). In [37], the problem of optimizing the excitation part of the boundary in lithotripsy is considered, with the initial-boundary value problem for the Westervelt equation (2.1) as the optimization constraint. This situation arises when excitation and self-focusing of high intensity ultrasound is performed by a piezo-mosaic (see fig. 1.2). Compared to the problem investigated there, the case of focusing by an acoustic lens presents us with several additional challenges. Not only a part of the domain boundary is optimized, but a subdomain which lies in the interior of the domain; this implies providing shape sensitivity analysis for an acoustic-acoustic interface problem. Insufficient spatial regularity of the primal (at most \( W^{1,\infty} \) in space) and the adjoint state (\( H^1 \) in space) on the whole domain does not allow for the shape derivative to be expressed in terms of the boundary integrals. However, as we saw in the previous chapter the state variable exhibits \( H^2 \)-regularity on each of the subdomains, provided that its gradient remains essentially bounded in space and time on the whole domain, which we will be able to use in our advantage.

Working with the state equation also implies handling the nonlinear damping term of the \( q \)-Laplace type, which is, in terms of shape sensitivity analysis, a nontrivial task as well.

Comment on the notation. To avoid confusion, it should be emphasized that while we use the dot notation for time differentiation and \( t \in [0, T] \) for the physical time variable, in this chapter \( \tau \in \mathbb{R} \) will be used for the artificial time variable to indicate varying (sub)domains. If \( \Omega_i \) is the initial shape of the lens, then \( \Omega_{+, \tau} \) will denote the perturbed lens obtained by moving points into the direction of some vector field \( h \) by some step length \( \tau \).

6.1.1 Analysis of the state equation. We recall that the local well-posedness result given in Proposition 4.14 holds. Since \( u - u_d \in L^2(0, T; L^2(\Omega)) \), the cost functional is well-defined.

6.1.2 Analysis of the adjoint problem. Since the state equation contains a \( q \)-Laplace type damping term, its linearization provides a significant challenge. The main difficulty in considering the linearized \( q \)-Laplacian lies in the need for an essential bound on the gradient of the solution \( u \); this is necessary for the linearized operator to be bounded.

We will consider the adjoint problem with the assumption that the weak solution \( u \) of (4.61) exhibits the following regularity:

\[ (\mathcal{H}_1) \quad u \in H^2(0, T; W^{1,\infty}(\Omega)) \cap X. \]

Note that the hypothesis is equivalent to assuming Lipschitz continuity of the acoustic pressure in space, i.e. \( u \in H^2(0, T; C^{0,1}(\Omega)) \cap X \) (cf. Theorem 3.2). Due to the embedding \( H^2(0, T) \rightarrow \)
To obtain a forward problem, let
\[ (6.4) \]
with \( (p, \tilde{p})|_{t=T} = (0, 0) \), where \( j(u) = (u - u_d)^2 \), \( X' = L^2(0, T; H^1(\Omega)) \) and we have used the notation
\[ G_u(Y) := |\nabla \dot{u}|^{q-1} Y + (q - 1)|\nabla \dot{u}|^{q-3}(\nabla \dot{u} \cdot Y)\nabla \dot{u} \]
for the linearized \( q \)-Laplace operator. The strong formulation of the adjoint problem then reads as:
\[ (6.5) \]
To obtain a forward problem, let \( t \in [0, T] \) and \( \tilde{p}(t) := p(T-t), \tilde{u}(t) := u(T-t) \) (cf. Lemma 3.17, [29]); then \( (\tilde{p}, \tilde{p})|_{t=0} = (0, 0) \), and we have the following problem for \( \tilde{p} \):

**Proposition 6.1.** (Local well-posedness of the adjoint problem) Let \( q \geq 1 \), \( q > d - 1 \), assumptions [5.5] on coefficients and hypothesis \((H_1)\) hold. For sufficiently small \( \tilde{m} \), final time \( T > 0 \), and \( \|\nabla \tilde{u}\|^q_{L^q(0, T; L^\infty(\Omega))} \leq \infty \), there exists a unique weak solution \( p \in X = C^1(0, T; L^2(\Omega)) \cap H^1(0, T; H^1_0(\Omega)) \) of \((6.4)\).

**Proof.** The well-posedness of the adjoint problem can be obtained through standard Galerking approximation in space, energy estimates and weak limits (cf. Section 7.2, [20]). We will focus
6.1. Problem formulation

Here on obtaining the crucial energy estimate. Testing \((6.5)\) with \(\zeta(\sigma) = \hat{p}(\sigma)\chi_{(0,\Omega)}\) (first in a discretized setting and then via weak limit transferred to the continuous one) yields

\[
\frac{1}{2} \left[ \int_{\Omega} (1 - 2\hat{\kappa}(\sigma)) (\hat{p}(\sigma))^2 \, dx + \int_{\Omega} \frac{1}{\hat{\kappa}} |\nabla \hat{p}(\sigma)|^2 \, dx \right]_0^t + \int_{\Omega} b(1 - \delta) |\nabla \hat{p}|^2 \, dx \, ds 
\]

\[
\geq - \int_{0}^{t} \int_{\Omega} \frac{k}{2} \hat{\kappa}(\sigma) \hat{p}^2 \, dx \, ds + \int_{0}^{t} \int_{\Omega} \hat{j}(\sigma) \hat{p} \, dx \, ds + \int_{0}^{t} \int_{\Omega} b \delta \varphi |\nabla \hat{p}| |\nabla \hat{p}| \, dx \, ds,
\]

where

\[
\varphi := |(\nabla \hat{u})^{q-1}| + (q - 1) |(\nabla \hat{u})^{q-3}| |\nabla \hat{u}|^2 + 2(q - 1) |\nabla \hat{u}|^{q-2} |\nabla \hat{u}| \leq C_\eta |\nabla \hat{u}|^{q-2} |\nabla \hat{u}|
\]

and we have used that \(\int_{0}^{t} \int_{\Omega} b \delta |\nabla \hat{u}|^{q-1} |\nabla \hat{p}|^2 \, dx \, ds \geq 0\) and \(\int_{0}^{t} \int_{\Omega} (q - 1) b \delta |\nabla \hat{u}|^{q-3} (\nabla \hat{p} \cdot \nabla \hat{u}) (\nabla \hat{u} \cdot \nabla \hat{p}) \geq 0\). By taking essential supremum with respect to \(t \in [0, T]\) in \((6.6)\) and employing

\[
\int_{0}^{T} \int_{\Omega} |\varphi| |\nabla \hat{p}| |\nabla \hat{p}| \, dx \, ds
\]

\[
\leq C_\eta |\nabla \hat{u}|^{q-2} \|L^\infty(0, T; L^\infty(\Omega))\| |\nabla \hat{u}|^2 \|L^2(0, T; L^2(\Omega))\| \left( \frac{1}{4e} \|\nabla \hat{p}\|^2 \|L^2(0, T; L^2(\Omega))\| + \varepsilon \|\nabla \hat{p}\|^2 \|L^2(0, T; L^2(\Omega))\| \right)
\]

we get the energy estimate for \(p\)

\[
\left( \frac{1}{4} (1 - a_0) - T \frac{\kappa}{\Delta} (C_{\Omega \Gamma, \mathcal{L}}^2)^2 \|\nabla u\|_{L^\infty(0, T; L^2(\Omega))} \right) \|\hat{p}\|^2 \|L^\infty(0, T; L^2(\Omega))\|
\]

\[
+ \left( \frac{1}{4e} - \frac{1}{4e} \delta \right) C_\eta |\nabla \hat{u}|^{q-2} \|L^\infty(0, T; L^\infty(\Omega))\| |\nabla \hat{u}|^2 \|L^2(0, T; L^2(\Omega))\|
\]

\[
+ \left( \frac{1}{2} b(1 - \delta) - \frac{\kappa}{\Delta} (C_{\Omega \Gamma, \mathcal{L}}^2)^2 \|\nabla u\|^2 \|L^2(0, T; L^2(\Omega))\| \right)
\]

\[
\leq \int_{0}^{T} \int_{\Omega} \hat{j}(\sigma) \hat{p} \, dx \, ds \leq 2 \|\hat{u} - \hat{u}_d\|_{L^2(0, T; L^2(\Omega))} \|\hat{p}\|^2 \|L^2(0, T; L^2(\Omega))\|,
\]

with \(1 - a_0 > 0\) and \(a_0\) is defined as in \((4.4)\), bounding away from zero factor \(1 - 2\hat{\kappa}\). This estimate holds under additional assumptions on smallness of \(T, \bar{m}\) and \(\|\nabla \hat{u}\|_{L^\infty(0, T; L^\infty(\Omega))}\) \(\|\nabla \hat{u}\|^2 \|L^2(0, T; L^\infty(\Omega))\|\):

\[
T \frac{\kappa}{\Delta} (C_{\Omega \Gamma, \mathcal{L}}^2)^2 C_P \bar{m} < \frac{1}{4} (1 - a_0),
\]

\[
\frac{1}{\varepsilon} \delta \bar{m} C_\eta \|\nabla \hat{u}\|^{q-2} \|L^\infty(0, T; L^\infty(\Omega))\| |\nabla \hat{u}|^2 \|L^2(0, T; L^\infty(\Omega))\| < \frac{1}{\delta},
\]

\[
\frac{\kappa}{\Delta} (C_{\Omega \Gamma, \mathcal{L}}^2)^2 C_P \bar{m} < \frac{1}{2} b(1 - \delta),
\]

and some sufficiently small \(\varepsilon > 0\). Here we have employed Poincaré’s inequality for functions in \(H_0^1(\Omega)\) (see Theorem 2.17, \([44]\)):

\[
\|\nabla \hat{u}\|_{L^\infty(0, T; L^2(\Omega))} \leq C_P \|\nabla \hat{u}\|_{L^2(0, T; L^2(\Omega))} \leq C_P \bar{m}, \quad \hat{u} \in \mathcal{W},
\]

\[
57.
\]
that $\Gamma + (\text{cf. [29]}).$ If the perturbed lens is given by $\Omega^\tau,$ it can be shown that there exists a fixed vector field $C^\tau,$ such that for $\tau > 0,$ the perturbation $e^\tau = \frac{1}{1 - 2k\hat{u}} \hat{\zeta},$ hence
\[
|\nabla \hat{\zeta}|_{L^2(\Omega)} = \left| \frac{\lambda}{1 - 2k\hat{u}} \nabla \hat{\zeta} + \frac{2k\lambda\hat{\zeta}}{(1 - 2k\hat{u})^2} \nabla \hat{u} \right|_{L^2(\Omega)} \\
\leq \frac{\lambda}{1 - a_0} |\nabla \hat{\zeta}|_{L^2(\Omega)} + \frac{2k\lambda}{(1 - a_0)^2} |\hat{\zeta}|_{L^2(\Omega)} |\nabla \hat{u}|_{L^2(\Omega)}.
\]
This further implies
\[
|\tilde{p}(t)|_{H^{-1}(\Omega)} \leq C(|\nabla \tilde{p}(t)|_{L^2(\Omega)} |\nabla \hat{u}(t)|_{L^2(\Omega)} |\nabla \hat{u}(t)|_{L^2(\Omega)}) \allowbreak + |\nabla \tilde{p}(t)|_{L^2(\Omega)} |\nabla \hat{u}(t)|_{L^2(\Omega)} |\hat{u}(t) - \hat{u}_d(t)|_{L^2(\Omega)},
\]
with $C = C(k, \lambda, \varrho, b, \delta, q) > 0.$ By squaring and integrating over $(0, T),$ we can achieve that $\tilde{p} \in L^2(0, T; H^{-1}(\Omega)).$

According to Theorem 3, Section 5.9, [20], since $\tilde{p} \in L^2(0, T; H^1_0(\Omega))$ and $\tilde{p} \in L^2(0, T; H^{-1}(\Omega)),$ it follows that $\tilde{p} \in C(0, T; L^2(\Omega)).$ The statement then comes from reversing the time transformation.

6.2 ELEMENTS OF SHAPE OPTIMIZATION

To be able to define the shape derivative of our cost functional, we need to introduce perturbations of the domain in a suitable way. To this end, we employ the so called perturbation of the identity transformations. We introduce a fixed vector field $h \in C^{1,1}(\Omega, \mathbb{R}^d)$ with $h|_{\partial \Omega} = 0,$ and a family of transformations $F_\tau : \Omega \rightarrow \mathbb{R}^d$

\[
F_\tau = id + \tau h, \quad \text{for } \tau \in \mathbb{R}.
\]
It can be shown that there exists $\tau_0 > 0,$ such that for $|\tau| < \tau_0,$ $F_\tau$ is a $C^{1,1}$-diffeomorphism (cf. 29). If the perturbed lens is given by $\Omega_{+\tau} = F_\tau(\Omega_\tau)$ and $\Gamma_\tau = F_\tau(\Gamma),$ then it follows that $\Gamma_\tau$ is Lipschitz continuous (see [23 Theorem 4.1]).
6.2. Elements of shape optimization

We mention that perturbations could be described alternatively by the velocity method as the flow determined by the initial-value problem

\[ \dot{\xi}(\tau) = h(\xi(\tau)) \]
\[ \xi(0) = x, \]

with \( F_{\tau}(x) = \xi(\tau; 0, x) \).

The Eulerian derivative of \( J \) at \( \Omega_{++} \) in the direction of the vector field \( h \) is defined as

\[ \frac{dJ}{du}(\Omega_{++}) h = \lim_{\tau \to 0} \frac{1}{\tau} (J(u_{\tau}, \Omega_{++}, \tau) - J(u, \Omega_{++})), \]

where \( u_{\tau} \) satisfies the state equation on the domain with the perturbed lens \( \Omega_{+-} \). The functional \( J \) is said to be shape differentiable at \( \Omega_{++} \) if \( \frac{dJ}{du}(\Omega_{++}) h \) exists for all \( h \in C^{1,1}(\Omega, \mathbb{R}^d) \) and defines a continuous linear functional on \( C^{1,1}(\Omega, \mathbb{R}^d) \).

We introduce the following notation:

\[ (6.8) \]

\[ I_{\tau} = \det(DF_{\tau}), \quad A_{\tau} = (DF_{\tau})^{-T}. \]

where \( DF_{\tau} \) is the Jacobian of the transformation \( F_{\tau} \).

**Lemma 6.2. [30]** For sufficiently small \( \tau_0 > 0 \), mapping \( F_{\tau} \) has the following properties:

\[ \tau \mapsto F_{\tau} \in C^{1}(-\tau_0, \tau_0; C^1(\Omega, \mathbb{R}^d)) \]
\[ \tau \mapsto A_{\tau} \in C(-\tau_0, \tau_0; C(\Omega, \mathbb{R}^{d \times d})) \]
\[ \tau \mapsto F_{\tau}^{-1} \in C(-\tau_0, \tau_0; C^1(\Omega, \mathbb{R}^d)) \]
\[ F_0 = \text{id} \]
\[ \frac{d}{d\tau} F_{\tau}|_{\tau=0} = h \]
\[ \frac{d}{d\tau} DF_{\tau}|_{\tau=0} = Dh \]
\[ \frac{d}{d\tau} I_{\tau}|_{\tau=0} = \text{div} h. \]

As a consequence of Lemma [6.2], there exist \( \alpha_0, \alpha_1 > 0 \) such that

\[ (6.9) \quad 0 < \alpha_0 \leq I_{\tau}(x) \leq \alpha_1, \quad \text{for } x \in \overline{\Omega}, \quad \tau \in [-\tau_0, \tau_0]. \]

Furthermore, there exist \( \beta_1, \beta_2 > 0 \) such that

\[ (6.10) \quad |A_{\tau}|_{L^\infty(\Omega)} \leq \beta_1, \quad |A_{\tau}^{-1}|_{L^\infty(\Omega)} \leq \beta_2, \quad \text{for } \tau \in [-\tau_0, \tau_0]. \]

We will often employ the following lemma which gives us the formula for the transformation of domain integrals:
Lemma 6.3. [56] Let \( \varphi \in L^1(\Omega_\tau) \), then \( \varphi \circ F_\tau \in L^1(\Omega) \) and
\[
\int_{\Omega_\tau} \varphi_\tau \ dx_\tau = \int_{\Omega} (\varphi \circ F_\tau) \ I_\tau \ dx.
\]

Finally, let us recall the well-known result obtained by Zolésio in 1979 which describes the structure of the shape gradient for sufficiently smooth domains:

Theorem 6.4. [14], Theorem 3.6, Corollary 1, Chapter 9 Let \( J \) be a real-valued shape functional. Assume that \( J \) is shape differentiable at some \( \Omega_+ \subset \mathbb{R}^d \) with boundary \( \Gamma \).

(i) The support of the shape gradient is contained in \( \Gamma \).

(ii) Let \( m \geq 0 \) be the smallest integer such that the map \( h \mapsto J(\Omega_+)h : C^{m+1}_c(\mathbb{R}^d, \mathbb{R}^d) \to \mathbb{R} \) is continuous. If the boundary \( \Gamma \) is of order \( C^{m+1} \), then there exists a scalar distribution \( g(\Gamma) \in \mathbb{R}^d \) with support in \( \Gamma \) such that
\[
dJ(\Omega_+)h = \langle g(\Gamma), \text{tr}_\Gamma h \cdot n \rangle_{C^m(\Gamma)}.
\]

(iii) If \( g \in L^1(\Gamma) \), then
\[
dJ(\Omega_+)h = \int_{\Gamma} gh \cdot n \ d\Gamma.
\]
The last formula is commonly called the Hadamard formula.

6.3 Existence of optimal shapes

From now on, for simplicity of exposition, we assume that all the coefficients in the state equation are piecewise constants, i.e.

\[
\begin{align*}
\lambda, k, \varrho, b, \delta & \in L^\infty(\Omega), \\
w_i := w|_{\Omega_i} & \text{ is constant, for } w \in \{b, \varrho, \lambda, \delta, k\}, i \in \{+, -\}, \\
w_i > 0 & \text{ for } w \in \{b, \varrho, \lambda\}, \ \delta_i \in (0, 1), \ k_i \in \mathbb{R}.
\end{align*}
\]

(5.7)

Note that now \( \omega = \min\{|\omega_+|, |\omega_-|\}, \ \overline{\omega} = \max\{|\omega_+|, |\omega_-|\}, \) where \( \omega \in \{b, \varrho, \lambda, \delta, k\} \).

We turn next to the question of existence of minimizers for the shape optimization problem 6.1 subject to the constraint (4.61), with the coefficients satisfying assumptions (5.7). The main idea of the proof is to employ the Bolzano-Weierstrass theorem for continuous functions on compact sets (for this approach see, for instance, [24 Section 3] and [41 Section 3.2]).

We recall that given two nonempty sets \( A \) and \( B \) in \( \mathbb{R}^d \), their Hausdorff distance is defined as
\[
d_H(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} |y - x|, \sup_{y \in B} \inf_{x \in A} |y - x|\}.
\]

The following compactness result holds:

Theorem 6.5. [24, Theorem 2] Let \( \Omega^*_n \) be a sequence in \( \mathcal{O}_{ad} \). Then there exists \( \Omega^*_+ \in \mathcal{O}_{ad} \) and a subsequence \( \Omega^*_{n_k} \) which converges to \( \Omega^*_+ \) in the sense of Hausdorff, and in the sense of characteristic functions. Additionally, \( \overline{\Omega}^+_{n_k} \) and \( \partial \Omega^*_{n_k} \) converge in the sense of Hausdorff towards \( \overline{\Omega}^+ \) and \( \partial \Omega^*_+ \), respectively.
Here the set of characteristic functions is defined as

\[ \text{Char}(\Omega) = \{ \chi_{\Omega_+} : \Omega_+ \subset \Omega \text{ is measurable } \land \chi_{\Omega_+} (1 - \chi_{\Omega_+}) = 0 \text{ a.e. in } \Omega \} \]

and convergence on this set, first of all, means pointwise almost everywhere convergence of functions, but due to Lebesgue’s dominated convergence theorem also convergence in \( L'(\Omega) \) for any \( r \in [1, \infty) \).

We first establish continuity of the mapping \( \Omega_+ \mapsto u \), where \( u \) solves the weak form of (6.2).

**Proposition 6.6.** Let \( q \geq 1, q > d - 1 \) and the assumptions (5.7) on the coefficients in (6.6) hold. Then the mapping \( \chi_{\Omega_+} \mapsto u \) is continuous from the set of characteristic functions \( \text{Char}(\Omega) \) to \( W^{1,\infty}(0, T; L^2(\Omega)) \cap W^{1,q+1}(0, T; W_0^{1,q+1}(\Omega)) \).

If additionally there exists \( \varepsilon > 0 \) such that the solution \( \tilde{u}^\varepsilon \) of (4.61) with \( \Omega_+ = \Omega_{+\varepsilon}^\varepsilon \) satisfies

\[ \|\tilde{u}^\varepsilon\|_{W^{1,q+1}(0, T; W^{1,q+1}(\Omega))} \leq C, \]

where \( C \) depends only on \( \Omega \) and the final time \( T \), then the mapping \( \chi_{\Omega_+} \mapsto u \) is even Hölder continuous at \( \chi_{\Omega_+} \).

**Proof.** Let \( \Omega_+^\varepsilon \) be a sequence converging to \( \Omega_{+\varepsilon}^\varepsilon \) in the sense of characteristic functions. By subtracting the weak forms (6.2) for \( u^n \) and \( u^\varepsilon \), corresponding to the domains with the lens regions \( \Omega_+^\varepsilon \) and \( \Omega_{+\varepsilon}^\varepsilon \) respectively, we get

\[
\sum_{i \in \{+, -\}} \int_0^T \int_\Omega \left\{ \frac{1}{\lambda_i} (1 - 2k_i u^n_t)(\dot{u}^n - \ddot{u}^\varepsilon) \phi - \frac{2k_i}{\lambda_i} (u^n - u^\varepsilon) \ddot{u}^\varepsilon \phi + \frac{1}{\bar{q}_i} \nabla (u^n - u^\varepsilon) \cdot \nabla \phi 
+ b_i (1 - \delta_i) \nabla (\dot{u}^n - \ddot{u}^\varepsilon) \cdot \nabla \phi + b_i \delta_i (|\nabla \dot{u}^n|^{q-1} \nabla \dot{u}^n - |\nabla \ddot{u}^\varepsilon|^{q-1} \nabla \ddot{u}^\varepsilon) \cdot \nabla \phi 
- \frac{2k_i}{\lambda_i} ((\dot{u}^n)^2 - (\ddot{u}^\varepsilon)^2) \phi \right\} \chi_{\Omega_+^\varepsilon} \ dx \ ds
+ b_i \delta_i |\nabla \ddot{u}^\varepsilon|^{q-1} \nabla \dot{u}^\varepsilon \cdot \nabla \phi - \frac{2k_i}{\lambda_i} (\ddot{u}^\varepsilon)^2 \phi \right\} \chi_{\Omega_{+\varepsilon}^\varepsilon} \ dx \ ds,
\]

with \( \Omega_+^\varepsilon = \Omega \setminus \Omega_{+\varepsilon}^\varepsilon, \Omega_{+\varepsilon}^\varepsilon = \Omega \setminus \Omega_{+\varepsilon}^\varepsilon \). Testing with \( \phi = \dot{u}^n - \ddot{u}^\varepsilon \) and employing inequality (3.12) for the difference of the \( q \)-Laplace terms then yields

\[
\frac{1}{2 \Lambda} (1 - 2k ||u^n||_{L^\infty(0,T; L^\infty(\Omega))}) \|\dot{u}^n - \ddot{u}^\varepsilon\|_{L^\infty(0,T; L^2(\Omega))}^2 + \frac{1}{2q} \|\nabla (u^n - u^\varepsilon)\|_{L^2(0,T; L^2(\Omega))}^2 
+ b(1 - \delta) \|\nabla (\dot{u}^n - \ddot{u}^\varepsilon)\|_{L^2(0,T; L^2(\Omega))}^2 + 2^{1-q} b\delta \|\nabla (\dot{u}^n - \ddot{u}^\varepsilon)\|_{L^{q+1}(0,T; L^{q+1}(\Omega))}^{q+1} 
\leq \frac{2F}{\Lambda} \|\ddot{u}^\varepsilon\|_{L^2(0,T; L^\infty(\Omega))} + \|\nabla u^n - \nabla u^\varepsilon\|_{L^2(0,T; L^2(\Omega))}^2 
+ (C_{H_1^{\varepsilon}})^2 \|u^n - u^\varepsilon\|_{L^\infty(0,T; H_1^{\varepsilon}(\Omega))} \|\ddot{u}^\varepsilon\|_{L^2(0,T; L^2(\Omega))} + \|\ddot{u}^\varepsilon\|_{L^2(0,T; H_1^{\varepsilon}(\Omega))} 
+ \sum_{i \in \{+, -\}} \int_0^T \int_\Omega \left\{ \frac{1}{\lambda_i} (1 - 2k_i u^n_t)(\dot{u}^n - \ddot{u}^\varepsilon) + \frac{1}{\bar{q}_i} \nabla u^n \cdot \nabla (\dot{u}^n - \ddot{u}^\varepsilon) \right\} \right. 
\]

\]

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we can estimate (6.12) as
\[ \frac{2k_i}{\lambda_i} (\dot{u}^n - \dot{u}^\sharp)^2 \right| \chi_{\Omega_i} - \chi_{\Omega^\sharp} \right| ds. \]

By employing the Sobolev embedding \( W_0^{1,q+1}(\Omega) \hookrightarrow L^\infty(\Omega) \):
\[ \| \dot{u}^\sharp \|_{L^2(0,T;L^\infty(\Omega))} \leq C_{W_0^{1,q+1},L^\infty} \| \dot{u}^\sharp \|_{L^2(0,T;W_0^{1,q+1}(\Omega))} \]
and then utilizing the energy estimate \([4.63]\) from Proposition \(4.14\) to estimate the terms \( \| \dot{u}^\sharp \|_{L^2(0,T;L^2(\Omega))} \) and \( \| \dot{u}^\sharp \|_{L^2(0,T;L^2(\Omega))} \), we conclude that for sufficiently small initial data the two first terms on the right hand side can be absorbed by the appropriate terms on the left hand side. For the remaining terms, we employ the following estimates:
\[ \int_0^T \int_\Omega \left\{ \frac{1}{\lambda_i} (1 - 2k_i u^n) \dot{u}^\sharp (\dot{u}^n - \dot{u}^\sharp) (\chi_{\Omega_i} - \chi_{\Omega^\sharp}) \right\} ds \]
\leq (1 + 2K \| u \|_{L^\infty(0,T;L^\infty(\Omega))}) \| \dot{u}^\sharp \|_{L^2(0,T;L^2(\Omega))} \| \dot{u}^n - \dot{u}^\sharp \|_{L^2(0,T;L^2(\Omega))} \| \dot{u}^\sharp \|_{L^2(0,T;L^2(\Omega))} \| \dot{u}^\sharp \|_{L^2(0,T;L^2(\Omega))} |\chi_{\Omega_i} - \chi_{\Omega^\sharp}|_{L^1(\Omega)},

and the estimate
\[ \int_0^T \int_\Omega \frac{1}{\theta} \nabla u^\sharp \cdot \nabla (\dot{u}^n - \dot{u}^\sharp) (\chi_{\Omega_i} - \chi_{\Omega^\sharp}) \right| ds \]
\leq \frac{1}{\theta} \| \nabla u^\sharp \|_{L^2(0,T;L^2(\Omega))} \| \nabla (\dot{u}^n - \dot{u}^\sharp) \|_{L^2(0,T;L^2(\Omega))} \| \dot{u}^\sharp \|_{L^2(0,T;L^2(\Omega))} \| \dot{u}^\sharp \|_{L^2(0,T;L^2(\Omega))} \| \dot{u}^\sharp \|_{L^2(0,T;L^2(\Omega))} \| \dot{u}^\sharp \|_{L^2(0,T;L^2(\Omega))} |\chi_{\Omega_i} - \chi_{\Omega^\sharp}|_{L^1(\Omega)},

with \( r_1 = \frac{2(q+1)}{q+2} \). An estimate analogous to the last one can be derived for the \( b_i(1 - \delta_i) \)-term where \( \nabla u^\sharp \) is replaced by \( \nabla \dot{u}^\sharp \). Thus all these terms on the right hand side tend to zero as \( n \to \infty \).

Finally, by Lebesgue’s dominated convergence theorem \([3.10]\) also the integral
\[ \int_0^T \int_\Omega b_i \delta_i | \nabla \dot{u}^\sharp |^{q-1} \nabla \dot{u}^\sharp \cdot \nabla (\dot{u}^n - \dot{u}^\sharp) (\chi_{\Omega_i} - \chi_{\Omega^\sharp}) \right| ds \]
goes to zero as \( n \to \infty \), since its integrand, due to the factor \( \chi_{\Omega_i} - \chi_{\Omega^\sharp} \), tends to zero pointwise a.e. and is bounded by the integrable function \( \frac{b_i}{\delta_i} | \nabla u^\sharp |^q (| \nabla u^n | + | \nabla u^\sharp |) \) whose integral is bounded by \( \int b_i | \nabla u^\sharp |^q_{L^q(0,T;L^{1+1/(q)}(\Omega))} \| \nabla u^n \|_{L^{q+1}(0,T;L^{s+1}(\Omega))} + \| \nabla u^\sharp \|_{L^{q+1}(0,T;L^{s+1}(\Omega))} \). This proves the assertion.

If we additionally assume that the solution of the state problem exhibits a slightly higher regularity in space than \( W^{1,q+1}(\Omega) \), namely that \( u \in W^{1,q+1}(0,T;W^{1,q+1+\varepsilon}(\Omega)) \) for some \( \varepsilon > 0 \), we can estimate \([6.12]\) as
\[ \int_0^T \int_\Omega b_i \delta_i | \nabla \dot{u}^\sharp |^{q-1} \nabla \dot{u}^\sharp \cdot \nabla (\dot{u}^n - \dot{u}^\sharp) (\chi_{\Omega_i} - \chi_{\Omega^\sharp}) \right| ds \]

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\[ \leq C|\nabla \dot{u}^2|_{L^{3+1}(0,T;L^{3+1}(\Omega))} \| \nabla (\dot{u}^2 - \dot{u}^2) \|_{L^{3+1}(0,T;L^{3+1}(\Omega))} |\chi_{\Omega^+_n} - \chi_{\Omega^+_n}|_{L^2(\Omega)}, \]

with \( r_2 = \frac{(1+\alpha)(1+\beta)}{q \epsilon} \). Altogether, we can then conclude that

\[
(1 - 2C\|w^n\|_{L^{\infty}(0,T;L^{\infty}(\Omega))}) \| \dot{u}^n - \dot{u}^2 \|_{L^{3+1}(0,T;L^{3+1}(\Omega))}^2 + \| \nabla (\dot{u}^n - \dot{u}^2) \|_{L^{3+1}(0,T;L^{3+1}(\Omega))}^2
\]

\[
+ \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega} \left\{ \frac{1}{\alpha_i} (1 - 2k_i \dot{u}^* \dot{u}) \dot{u}^* \phi + \frac{1}{b_i} \nabla u^* \cdot \nabla \phi + b_i (1 - \delta_i) \nabla \dot{u}^* \cdot \nabla \phi \right\} dx \, ds.
\]

for sufficiently large \( C > 0 \) independent of \( n \), which implies H"older continuity of the mapping \( \chi_{\Omega^+_n} \rightarrow u \).

Now let \( \Omega^+_n \) be a minimizing sequence for (6.1), (4.61). Due to Theorem 6.5, there exists a subsequence, which, for brevity, we still denote \( \Omega^+_n \), that converges to some \( \Omega^*_+ \in \mathcal{O}_{ad} \). By extracting another sequence, we may as well assume that \( \Omega^+_n \rightarrow \Omega^*_+ \) in the sense of characteristic functions.

Let us denote by \( u^n \) the weak solution corresponding to the domain where the lens region is given by \( \Omega^*_+ \). We know then that \( u^n \) satisfies the estimate (4.63), where \( u \) is the strong solution of (6.1) in the domain whose lens region is given by \( \Omega^*_+ \).

\[ u^n \rightharpoonup u^* \text{ in } H^2(0,T;L^2(\Omega)), \]

\[ \nabla \dot{u}^n \rightharpoonup \nabla \dot{u}^* \text{ in } L^{q+1}(0,T;L^{q+1}(\Omega)). \]

Due to the embedding \( H^2(0,T) \hookrightarrow C^1(0,T) \), this further implies that \( u(t) \rightharpoonup u^*(t) \) and \( \dot{u}(t) \rightharpoonup \dot{u}^*(t) \) in \( L^2(\Omega) \) for all \( t \in [0,T] \). Therefore, \( (u^*, \dot{u}^*)|_{t=0} = (u_0, u_1) \).

It remains to show that \( u^* \) solves the state problem on the domain whose lens region is given by \( \Omega^*_+ \).

**Proposition 6.7.** Let the assumptions of Proposition 6.6 be satisfied. Let \( \Omega^+_n \) be a minimizing sequence for the shape optimization problem (6.1), (4.61), and let \( \Omega^*_+ \) be an accumulation point of this sequence in accordance with Theorem 6.5. Then the sequence \( u^n \) corresponding to the domain with the lens region \( \Omega^+_n \) converges strongly to \( u^* \) in \( W^{1,\infty}(0,T;L^2(\Omega)) \cap W^{1,q+1}(0,T;W_0^{1,q+1}(\Omega)) \), where \( u^* \) is the solution of (4.61) in the domain whose lens region is given by \( \Omega^*_+ \).

**Proof.** In order to see that \( u^* \) is the weak solution of the state problem in the domain where the lens region is given by \( \Omega^*_+ \), note that due to the fact that \( u^n \) solves (4.61) with lens region \( \Omega^*_n \) and using integration by parts in the first term on the right hand side

\[
\sum_{i \in \{+, -\}} \int_0^T \int_{\Omega} \left\{ \frac{1}{\alpha_i} (1 - 2k_i u^*) \dot{u}^* \phi + \frac{1}{b_i} \nabla u^* \cdot \nabla \phi + b_i (1 - \delta_i) \nabla \dot{u}^* \cdot \nabla \phi \right\} dx \, ds
\]

\[
+ b_i \delta_i |\nabla \dot{u}^*|^{q-1} \dot{u}^* \cdot \nabla \phi - \frac{2k_i}{\alpha_i} (\dot{u}^*)^2 \phi \chi_{\Omega^*_n} \right\} dx \, ds.
\]
\[ \sum_{i \in \{+,-\}} \int_0^T \int_{\Omega} \left\{ \frac{1}{\lambda_i} (1 - 2k_i u^*)(\hat{u}^* - \hat{u}^n) + \frac{1}{\bar{\varrho}_i} \nabla (u^* - u^n) \cdot \nabla \phi \right\} \chi_{\Omega^*} \, dx \, ds \]

\[ + b_i (1 - \delta_i) \nabla (u^* - u^n) \cdot \nabla \phi + b_i \delta_i \| \nabla \hat{u}^* \| q^{-1} \nabla \hat{u}^* - \| \nabla \hat{u}^n \| q^{-1} \nabla \hat{u}^n \cdot \nabla \phi \]

\[ - \frac{2k_i}{\lambda_i} (\hat{u}^* - \hat{u}^n) \nabla \phi \} \chi_{\Omega^*} \, dx \, ds \]

\[ + \sum_{i \in \{+,-\}} \int_0^T \int_{\Omega} \left\{ \frac{1}{\lambda_i} (1 - 2k_i u^n) \hat{u}^n \nabla \phi + \frac{1}{\varrho_i} \nabla u^n \cdot \nabla \phi + b_i (1 - \delta_i) \nabla \hat{u}^n \cdot \nabla \phi \right\} \chi_{\Omega^*} \, dx \, ds, \]

for all \( \phi \in C^\infty(0,T; C_0^\infty(\Omega)) \), \( \phi(T) = 0 \). The difference of the \( q \)-Laplace terms on the right hand side can be estimated with the help of inequality \( 3.11 \) (with \( \eta = 0 \)) as follows:

\[ \int_0^T \int_{\Omega} b_i \delta_i (|\nabla \hat{u}^*| q^{-1} |\nabla \hat{u}^* - |\nabla \hat{u}^n| q^{-1} |\nabla \hat{u}^n|) \cdot \nabla \phi \chi_{\Omega^*} \, dx \, ds \]

\[ \leq C_q b_i \delta_i \int_0^T \int_{\Omega} |\nabla (\hat{u}^* - \hat{u}^n) | | | \nabla \hat{u}^*| q^{-1} + |\nabla \hat{u}^n| q^{-1} | | \nabla \phi | | \chi_{\Omega^*} \, dx \, ds \]

\[ \leq C_q b_i \delta_i \| \nabla (\hat{u}^* - \hat{u}^n) \| _{L^q(0,T;L^q(\Omega))} \left( \| \nabla \hat{u}^* \| _{L^q(0,T;L^q(\Omega))} \right) \]

\[ + \| \nabla \hat{u}^n \| _{L^q(0,T;L^q(\Omega))} \| \nabla \phi \| _{L^q(0,T;L^q(\Omega))}. \]

The remaining terms can be estimated analogously to the estimates in the proof of Proposition 6.6 from which it then follows that the right hand side in (6.13) tends to zero as \( n \to \infty \).

**Theorem 6.8.** Let \( q \geq 1 \), \( q > d - 1 \) and the assumptions (5.7) on the coefficients in (5.6) hold. Then the shape optimization problem (6.1) subject to (4.61) has a solution.

**Proof.** Let us define the reduced cost functional \( \hat{J} : \mathcal{O}_{ad} \to \mathbb{R} \):

\[ \hat{J}(\Omega^+) = J(u(\Omega^+), \Omega^+). \]

Let \( \Omega^+_n \to \Omega^+ \) as \( n \to \infty \). It can be shown (cf. Proposition 3.3) that

\[ \| J(u^*, \Omega^+) - J(u^n, \Omega^+_n) \| \leq \| u^* - u^n \| _{L^q(0,T;L^q(\Omega))} \| \nabla u^* + u^n - 2u_d \| _{L^q(0,T;L^q(\Omega))}. \]

Since the term \( \| u^* + u^n - 2u_d \| _{L^q(0,T;L^q(\Omega))} \) is uniformly bounded due to (4.63), by employing Proposition 6.6 we achieve that the right hand side tends to zero as \( n \to \infty \). Therefore the cost functional \( \hat{J} \) is continuous on \( \mathcal{O}_{ad} \). According to the Bolzano-Weierstrass theorem 3.12 since \( \mathcal{O}_{ad} \) is compact, \( \hat{J} \) attains a global minimum on \( \mathcal{O}_{ad} \). 

**6.4 State equation on the domain with perturbed lens**

Let us introduce, for some Lipschitz domain \( \Omega^+ \in \mathcal{O}_{ad} \), the operator \( E(\cdot, \Omega^+) : \mathcal{W} \to \hat{X}^* \) by

\[ \langle E(u, \Omega^+), \phi \rangle_{\hat{X}^*, \hat{X}} = \sum_{i \in \{+,-\}} \int_0^T \int_{\Omega^+_i} \left\{ \frac{1}{\lambda_i} (1 - 2k_i u) \hat{u}^i \phi + \frac{1}{\bar{\varrho}_i} \nabla u \cdot \nabla \phi \right\} \chi_{\Omega^*} \, dx \, ds. \]
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Fix $\tau \in (-\tau_0, \tau_0)$. Proposition 6.9 guarantees that $E(u, \Omega_{+,\tau}) = 0$ has a unique solution, which we denote $u_\tau : \Omega \to \mathbb{R}$. We transport $u_\tau$ back to the domain with the fixed lens $\Omega_+$ by defining $u^\tau : \Omega \to \mathbb{R}$ as

$$u^\tau = u_\tau \circ F_{\tau}.$$ 

Differentiability of $h$ implies that $u^\tau \in X$. We further have

$$\langle E(u_\tau, \Omega_{+,\tau}), \phi_\tau \rangle_{X^*,X} = \sum_{i \in \{+,-\}} \int_0^T \int_{\Omega_{i,\tau}} \left\{ \frac{1}{\lambda_i} (1 - 2k_i u_\tau) \tilde{u}_\tau \phi_\tau + \frac{1}{\vartheta_i} \nabla u_\tau \cdot \nabla \phi_\tau + b_i (1 - \delta_i) \nabla \tilde{u}_\tau \cdot \nabla \phi_\tau + b_i \delta_i |\nabla \tilde{u}_\tau|^{q-1} \nabla \tilde{u}_\tau \cdot \nabla \phi_\tau - \frac{2k_i}{\lambda_i} (\tilde{u}_\tau)^2 \phi_\tau \right\} dx_{\tau} ds$$

(6.15)

for any $\phi_\tau \in \tilde{X}_{\tau} = \tilde{X}$, with $\Omega_{-,\tau} = \Omega \setminus \overline{\Omega}_{+,\tau}$, where we have used Lemma 6.3 and the fact that

$$\nabla u_\tau = A_{\tau} \nabla u^\tau \circ F_{\tau}^{-1}.$$ 

Therefore, for sufficiently small $|\tau|$, $u^\tau$ uniquely satisfies an equation on the domain with the fixed lens:

(6.16) \[ \tilde{E}(u^\tau, \tau) = 0. \]

Since $F_0 = id$, we have that $u^0 = u$ and $\tilde{E}(u, 0) = E(u, \Omega_+)$. 

6.4.1 Continuity of the state with respect to domain perturbations. We will now focus our attention on the speed of convergence of $u^\tau$ to $u$ as $\tau \to 0$ and prove two properties which together correspond to hypothesis (H2) in [30] (see also [29, Proposition 3.1]).

We begin with the question of uniform boundedness of $u^\tau$ with respect to $\tau \in (-\tau_0, \tau_0)$. Since

$$\|u^\tau\|_{L^\infty(0,T;L^\infty(\Omega))} = \|u_\tau \circ F_{\tau}\|_{L^\infty(0,T;L^\infty(\Omega))} \leq \|u_\tau\|_{L^\infty(0,T;L^\infty(\Omega))},$$

we also have that

(6.17) \[ 1 - a_0 < \|1 - 2k u^\tau\|_{L^\infty(0,T;L^\infty(\Omega))} < 1 + a_0, \]

where $a_0$ is defined as in (1.4).

**Proposition 6.9.** Let $q \geq 1$, $q > d - 1$ and assumptions (5.7) hold. Solutions $u^\tau$ of (6.16) are uniformly bounded in $W^{1,\infty}(0,T;L^2(\Omega)) \cap W^{1,q+1}(0,T;W^{1,q+1}_0(\Omega))$ for $\tau \in (-\tau_0, \tau_0)$. 

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Proof. It can be shown (similarly to [29, Lemma 3.3]) that

\[ \|\nabla u^\tau\|_{L^{q+1}(0,T;L^{q+1}(\Omega))} \leq \frac{1 + \tau_0 |Dh|_{L^\infty(\Omega)}}{\alpha_0^{1/(q+1)}} \|\nabla u_{\tau^+}\|_{L^{q+1}(0,T;L^{q+1}(\Omega))}, \]

with \( \alpha_0 \) as in (6.9). This implies that, for \( u_\tau \in \mathcal{W} \), we can estimate

\[ (6.18) \quad \|\nabla u^\tau\|_{L^{q+1}(0,T;L^{q+1}(\Omega))} \leq \frac{1 + \tau_0 |Dh|_{L^\infty(\Omega)}}{\alpha_0^{1/(q+1)}} \mathcal{M}. \]

Testing (6.16) with \( \phi^\tau = u^\tau \chi_{[0,t]} \in L^2(0,T;W_0^{1,q+1}(\Omega)) \), with \( \tilde{E} \) as in (6.15), yields

\[
\begin{align*}
\frac{1}{2} \left[ \int_\Omega (1 - 2k u^\tau)(\dot{u}^\tau)^2 I_{\tau^+} \, dx \right]_0^T + \frac{1}{2} \left[ \int_\Omega |A_\tau \nabla u^\tau|^2 I_{\tau^+} \, dx \right]_0^T \\
+ \int_0^T \int_\Omega b(1 - \delta) |A_\tau \nabla u^\tau|^2 I_{\tau^+} \, dx \, ds + \int_0^T \int_\Omega b\delta |A_\tau \nabla u^\tau|^2 (\dot{u}^\tau)^2 I_{\tau^+} \, dx \, ds \\
= \int_0^T \int_\Omega \frac{k}{\lambda} (\dot{u}^\tau)^3 I_{\tau^+} \, dx \, ds.
\end{align*}
\]

By taking the supremum over \( t \in [0,T] \) and by utilizing the uniform boundedness properties (6.9), (6.10) and estimates (6.17) and (6.18), we find that

\[
\begin{align*}
\frac{1}{\Lambda} \alpha_1 \|u^\tau\|_{L^2(0,T;L^2(\Omega))} \sqrt{T} \|\dot{u}^\tau\|_{L^\infty(0,T;L^2(\Omega))} &+ \frac{1}{2\Lambda} (1 + \alpha_0) \alpha_1 |u_1|_{L^2(\Omega)} \\
&\leq \frac{1}{\Lambda} \alpha_1 \|u^\tau\|_{L^2(0,T;L^2(\Omega))} + \frac{1}{2\Lambda} (1 + \alpha_0) \alpha_1 |u_1|_{L^2(\Omega)}
\end{align*}
\]

(6.19)

\[
\begin{align*}
&\leq \frac{1}{\Lambda} \alpha_1 \|u^\tau\|_{L^2(0,T;L^2(\Omega))} + \frac{1}{2\Lambda} (1 + \alpha_0) \alpha_1 |u_1|_{L^2(\Omega)} \\
&+ \frac{1}{2\Lambda} \alpha_1 \|u^\tau\|_{L^2(0,T;L^2(\Omega))} + \frac{1}{2\Lambda} (1 + \alpha_0) \alpha_1 |u_1|_{L^2(\Omega)}
\end{align*}
\]

From here, for sufficiently small \( \mathcal{M} \) and \( T \), we can achieve that the first term on the right hand side gets absorbed by the appropriate term on the left hand side. Thus we have uniform boundedness of \( u^\tau \) in \( W^{1,\infty}(0,T;L^2(\Omega)) \cap W^{1,q+1}(0,T;W_0^{1,q+1}(\Omega)) \), \( |\tau| < \tau_0 \).

The Hölder continuity of \( u \) with respect to domain perturbations is established by our next result.

**Proposition 6.10.** Let \( q \geq 1 \), \( q > d - 1 \) and let assumptions (5.7) hold. Then

\[
\lim_{\tau \to 0} \frac{1}{\tau} \left( \|\dot{u}^\tau - \dot{u}\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla (u^\tau - u)\|_{L^\infty(0,T;L^2(\Omega))} \right.
\]

\[
+ \|\nabla (u^\tau - u)\|_{L^2(0,T;L^2(\Omega))} + \|\nabla (u^\tau - u)\|_{L^{q+1}(0,T;L^{q+1}(\Omega))} \right) = 0.
\]

(6.20)
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Proof. Note that the difference \( v^r = u - u^r \) satisfies

\[
\begin{align*}
\sum_{i \in \{+,\,-\}} \int_0^T \int_{\Omega_i} \left\{ \frac{1}{\lambda_i} (1 - 2k_i u_i^r) \ddot{v}_i^r \phi_i I_r - \frac{2k_i}{\lambda_i} \dot{v}_i^r \ddot{u}_i I_r + \frac{1}{\theta_i} \nabla v_i^r \cdot \nabla \phi_i \\
+ b_i (1 - \delta_i) \nabla \dot{v}_i^r \cdot \nabla \phi_i + b_i \delta_i (|\nabla \dot{u}_i|^q - |\nabla \dot{u}_i|^q - 1) \nabla \phi_i \\
- \frac{2k_i}{\lambda_i} (\dot{u}_i + \dot{u}_i^r) \ddot{v}_i^r \phi_i \right\} \, dx \, ds \\
= (f_+, \phi_+) \ddot{x}_+^r, \ddot{x}_-^r + (f_-, \phi_-) \ddot{x}_-^r, \ddot{x}_+^r,
\end{align*}
\]

for all \( \phi \in \tilde{X} \), with the two terms on the right hand side given by

\[
\begin{align*}
\langle f_i, \phi_i \rangle_{\dddot{x}_i^r, \dddot{x}_i^r} \\
= \int_0^T \int_{\Omega_i} \left\{ \frac{1}{\lambda_i} (I_r - 1) (1 - 2k_i u_i) \ddot{u}_i \phi_i \phi_i + \frac{1}{\theta_i} ((I_r - 1) A_r \nabla u_i^r \cdot A_r \nabla \phi_i \\
+ (A_r - I) \nabla u_i^r \cdot A_r \nabla \phi_i + \nabla u_i^r \cdot (A_r - I) \nabla \phi_i) \\
+ b_i (1 - \delta_i) ((I_r - 1) A_r \nabla \dot{u}_i^r \cdot A_r \nabla \phi_i + (A_r - I) \nabla \dot{u}_i^r \cdot A_r \nabla \phi_i + \nabla \dot{u}_i^r \cdot (A_r - I) \nabla \phi_i) \\
+ b_i \delta_i ((A_r \nabla \dot{u}_i^r |^{q-1} A_r \nabla \ddot{u}_i^r - |\nabla \dot{u}_i^r|^{q-1} \nabla \ddot{u}_i^r) \cdot \nabla \phi_i + |A_r \nabla \dot{u}_i^r |^{q-1} A_r \nabla \dot{u}_i^r \cdot (A_r - I) \nabla \phi_i \\
+ (I_r - 1) A_r \nabla \dot{u}_i^r |^{q-1} A_r \nabla \dot{u}_i^r \cdot A_r \nabla \phi_i) - (I_r - 1) \frac{2k_i}{\lambda_i} (\dot{u}_i^r) \phi_i \right\} \, dx \, ds,
\end{align*}
\]

where \( \dddot{x}_i = L^2(0,T;W^{1,q+1}(\Omega_i)) \). Since \( \phi_i = \tilde{v}_i \chi_{\{0\}} \in \tilde{X} \), we can use it as test functions in (6.21), which together with the uniform boundedness of \( I_r \) results in

\[
\sum_{i \in \{+,\,-\}} \left\{ \frac{1}{2} \alpha_0 \int_{\Omega_i} \frac{1}{\lambda_i} (1 - 2k_i u_i^r) (\tilde{v}_i^r)^2 \, dx + \frac{1}{2} \int_{\Omega_i} \frac{1}{\theta_i} |\nabla \tilde{v}_i^r|^2 \, dx \right\} \\
+ \int_0^T \int_{\Omega_i} b_i (1 - \delta_i) |\nabla \tilde{v}_i^r|^2 \, dx \, ds + \int_0^T \int_{\Omega_i} b_i \delta_i 2^{1-q} |\nabla \tilde{u}_i^r|^{q+1} \, dx \, ds \leq \sum_{i \in \{+,\,-\}} \left\{ \frac{K}{\lambda_i} \left[ \alpha_1 \|\tilde{v}_i^r\|_{L^2(0,T;L^\infty(\Omega_i))} \|\dot{\tilde{v}}_i^r\|_{L^\infty(0,T;L^2(\Omega_i))} \\
+ 2(C_{1,1}^2 \|\tilde{u}_i\|_{L^2(0,T;L^\infty(\Omega_i))} \|\tilde{u}_i\|_{L^\infty(0,T;H^1(\Omega_i))} \|\tilde{v}_i\|_{L^2(0,T;H^1(\Omega_i))} \\
+ 2(\|\tilde{u}_i^r\|_{L^2(0,T;L^\infty(\Omega_i))} + \|\tilde{u}_i\|_{L^2(0,T;L^\infty(\Omega_i))}) \|\tilde{v}_i^r\|_{L^\infty(0,T;L^2(\Omega_i))} \right] + \|f_i, \tilde{v}_i^r\|_{\tilde{x}_i^r, \tilde{x}_i^r} \right\}.
\]

Here we have also utilized the fact that

\[
(|\nabla \dot{u}_i|^q - 1) \nabla \dot{u}_i - |\nabla \dot{u}_i|^q - 1) \nabla \ddot{u}_i \geq 2^{1-q} |\nabla (\dot{u}_i - \ddot{u}_i)|^{q+1},
\]

which follows from inequality (3.12).

We can employ

\[
\|\tilde{u}_i^r\|_{L^2(0,T;L^\infty(\Omega_i))} \leq T^{\frac{q}{2(q+1)}} C_{W^{1,q+1},L} \|\tilde{u}_i^r\|_{L^{q+1}(0,T;W^{1,q+1}(\Omega_i))},
\]

and the same inequality with \( \tilde{u}_i \) instead of \( \tilde{u}_i^r \), as well as

\[
\|v_i\|_{L^\infty(0,T;H^1(\Omega_i))} \leq \sqrt{T} \|\tilde{v}_i\|_{L^2(0,T;H^1(\Omega_i))},
\]

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We can therefore estimate the terms on the right hand side in (6.23) as

\[
\sum_{i \in \{+, -\}} (\| \tilde{v}_i^T \|_{L^2(0,T;L^2(\Omega))}^2 + \| \nabla v^T \|_{L^2(0,T;L^2(\Omega))}^2 + \| \nabla \tilde{v}_i^T \|_{L^2(0,T;L^2(\Omega))}^2)
\]

for some sufficiently large \( C > 0 \) which does not depend on \( \tau \). By employing the inequality (3.11) (with \( \eta = 0 \)) we obtain

\[
| (f_i, \tilde{v}_i^T)_{\dot{X}_+^*, \bar{X}_-^*} | \leq \frac{1}{\lambda} |I_\tau - 1|_{L^\infty(\Omega)}(1 + a_0) \sup_{\tau \in (\overline{\tau}, 0)} \| \tilde{v}_i \|_{L^2(0,T;L^2(\Omega))} \| \tilde{v}_i^T \|_{L^2(0,T;L^2(\Omega))} + (1 + \beta_1) \| \nabla v^T \|_{L^2(0,T;L^2(\Omega))}
\]

\[
+ \frac{\bar{\delta}(1 + \beta_2)}{\lambda} \| \nabla \tilde{v}_i^T \|_{L^2(0,T;L^2(\Omega))} \| \nabla \tilde{v}_i^T \|_{L^2(0,T;L^2(\Omega))} + \frac{\bar{\delta} q (A_\tau - I)_{L^\infty(\Omega)}(1 + |A_\tau - I|_{L^\infty(\Omega)}) + (A_\tau - I)_{L^\infty(\Omega)} \beta_1^q}{\lambda} \| \nabla \tilde{v}_i^T \|_{L^2(0,T;L^2(\Omega))} + \frac{2\bar{\delta}}{\lambda} (C_{1\Omega}^{H^1(\Omega)})^2 \| \tilde{v}_i^T \|_{L^2(0,T;L^2(\Omega))} \| \tilde{v}_i^T \|_{L^2(0,T;L^2(\Omega))}
\]

where \( a_0 \) is defined as in (4.4). By inserting this into the estimate (6.23) and by employing the uniform boundedness result from Proposition 6.9 and properties of the mapping \( F_\tau \) from Lemma 6.2 and Young’s inequality, we can conclude that

\[
\lim_{\tau \rightarrow 0} \sum_{i \in \{+, -\}} (\| \tilde{v}_i^T \|_{L^2(0,T;L^2(\Omega))}^2 + \| \nabla v^T \|_{L^2(0,T;L^2(\Omega))}^2 + \| \nabla \tilde{v}_i^T \|_{L^2(0,T;L^2(\Omega))}^2)
\]

\[
+ \| \nabla \tilde{v}_i^T \|_{L^2(0,T;L^2(\Omega))}^2 = 0.
\]

In order words, we know that \( \lim_{\tau \rightarrow 0} \| \tilde{v}_i^T \| = u \in X \). To obtain the statement of the Proposition, we divide (6.23) by \( \tau \), and then it remains to show that

\[
\lim_{\tau \rightarrow 0} \sup_{\tau \in (\overline{\tau}, 0)} \| (f_+, \tilde{v}_i^T)_{\dot{X}_+^*, \bar{X}_-^*} \| + \| (f_-, \tilde{v}_i^T)_{\dot{X}_+^*, \bar{X}_-^*} \| = 0.
\]

This now follows from the estimate (6.25), Lemma 6.2 Proposition 6.9 and (6.26). □

If we assume higher spatial regularity of \( u \), we can even obtain Lipschitz continuity with respect to domain perturbations.
6.4. State equation on the domain with perturbed lens

**Proposition 6.11.** Let $q \geq 1$, $q > d - 1$ and let assumptions \[5,7\] hold. Assume that the solution $u$ of (4.61) satisfies

\[ \| \nabla \hat{u} \|_{L^2(0,T;L^2(\Omega))} \leq \hat{C}, \]

where $\hat{C}$ depends only on $\Omega$ and the final time $T$. Then

\[ \frac{1}{\tau}(\| \hat{u}^T - \hat{u} \|_{L^\infty(0,T;L^2(\Omega))} + \| \nabla (\hat{u}^T - u) \|_{L^\infty(0,T;L^2(\Omega))} + \| \nabla (\hat{u}^T - \hat{u}) \|_{L^2(0,T;L^2(\Omega))}) \leq C \]

for all $\tau \in (-\tau_0, \tau_0)$, $\tau \neq 0$, where $C$ does not depend on $\tau$.

**Proof.** We can rewrite the norm on $\hat{X} = C^1(0,T;L^2(\Omega)) \cap H^1(0,T;H^1_0(\Omega))$ (cf. Proposition 6.1) as

\[ \| u \|_{\hat{X}} := \left( \sum_{i \in \{+, -\}} \| u_i \|_{\hat{X}_i}^2 \right)^{1/2}, \]

where

\[ \| u \|_{\hat{X}_i} := \left( \| \hat{u}_i \|_{L^\infty(0,T;L^2(\Omega_i))} + \| \nabla \hat{u}_i \|_{L^\infty(0,T;L^2(\Omega_i))} + \| \nabla \hat{u}_i \|_{L^2(0,T;L^2(\Omega_i))} \right)^{1/2}. \]

By employing assumption \[6.27\], we can modify estimate \[6.25\] by changing the second to last line as follows:

\[ \| v^T \|_{\hat{X}} \leq \sum_{i \in \{+, -\}} |(f_{iT}, \hat{v}_i^T)_{\hat{X}_i^*, \hat{X}_i}| \leq C(\| \hat{I}_T - 1 \|_{L^\infty(\Omega_i)} + \| A_T - I \|_{L^\infty(\Omega_i)}) \| v^T \|_{\hat{X}}. \]

This further implies that

\[ \| v^T \|_{\hat{X}} \leq C(\| \hat{I}_T - 1 \|_{L^\infty(\Omega_i)} + \| A_T - I \|_{L^\infty(\Omega_i)}) \| v^T \|_{\hat{X}}, \]

where $C > 0$ does not depend on $\tau$, from which we can conclude that

\[ \| v^T \|_{\hat{X}} \leq C(\| \hat{I}_T - 1 \|_{L^\infty(\Omega_i)} + \| A_T - I \|_{L^\infty(\Omega_i)}). \]

The assertion then follows by applying Lemma 6.2.
6.5 Auxiliary results

In order to calculate the shape derivative of our cost functional, we will need to employ the two forthcoming propositions stating certain smoothness properties of \( E \) and \( \tilde{E} \), defined as in (6.14), (6.15). The assertions correspond to hypotheses (H4) and (H3) in [30]. Note that, since

\[
|\nabla u^*|_{L^\infty(0,T;L^\infty(\Omega))} \leq C,
\]

we also know that \( |\nabla u^*|_{L^\infty(0,T;L^\infty(\Omega))} \) is uniformly bounded for \( |\tau| < \tau_0 \). Then also condition (6.27) holds.

**Proposition 6.12.** Assume that the coefficients in the state equation satisfy (5.11) and \( q > 2 \). Let hypotheses (H1) and (H2) hold. Then

\[
\lim_{t \to 0} \frac{1}{\tau} \langle \tilde{E}(u^*, \tau) - \tilde{E}(u, \tau) - (E(u^*, \Omega) - E(u, \Omega)), p \rangle_{\bar{X}, \bar{X}} = 0
\]

holds for the adjoint state \( p \).

Proof. We begin by calculating the difference

\[
\frac{1}{\tau} \langle \tilde{E}(u^*, \tau) - \tilde{E}(u, \tau) - (E(u^*, \Omega) - E(u, \Omega)), p \rangle_{\bar{X}, \bar{X}} = \frac{1}{\tau} \sum_{i \in \{+, -\}} (I_i + II_i + III_i + IV_i),
\]

where we use the following notation

\[
I_i = \int_0^T \int_{\Omega_i} \frac{1}{\lambda_i} ((1 - 2k_i u_i^*)(\tilde{u}_i^* - \tilde{u}_i) - 2k_i (u_i^* - u_i) \tilde{u}_i)(I_i - 1)p_i \, dx \, ds,
\]

\[
II_i = \int_0^T \int_{\Omega_i} \left\{ \left( A_{\tau} - I \right) \left( \frac{1}{\rho_i} \nabla(u_i^* - u_i) + b_i(1 - \delta_i) A_{\tau}(\tilde{u}_i^* - \tilde{u}_i) \right) A_{\tau} \nabla p_i \\
+ (I_i - 1) \left( \frac{1}{\rho_i} A_{\tau}(u_i^* - u_i) + b_i(1 - \delta_i) A_{\tau}(\tilde{u}_i^* - \tilde{u}_i) \right) \cdot A_{\tau} \nabla p_i \\
+ \left( \frac{1}{\rho_i} \nabla(u_i^* - u_i) + b_i(1 - \delta_i) \nabla(\tilde{u}_i^* - \tilde{u}_i) \right) \cdot (A_{\tau} - I) \nabla p_i \right\} \, dx \, ds,
\]

\[
III_i = \int_0^T \int_{\Omega_i} b_i \delta_i \left\{ \left( A_{\tau} \nabla \tilde{u}_i^* \right|_{\rho - 1} A_{\tau} \nabla \tilde{u}_i^* - |A_{\tau} \nabla \tilde{u}_i|_{\rho - 1} A_{\tau} \nabla \tilde{u}_i \right) \cdot A_{\tau} \nabla p_i I_i \right\} \, dx \, ds.
\]
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\[- \left( |\nabla \tilde{u}_i|^q \nabla \tilde{u}_i - |\nabla \tilde{u}_i|^{q-1} \nabla \tilde{u}_i \right) \cdot \nabla p_i \right) \, dx \, ds,

IV_i = - \int_0^T \int_{\Omega_i} \frac{2k_i}{\Delta} (\tilde{u}_i^T - \tilde{u}_i) (\tilde{u}_i^T + \tilde{u}_i) (I_\tau - 1)p_i \, dx \, ds,

i \in \{+, -\}. Thanks to hypothesis (H_1), we can estimate the first integral as

\[|I_i| \leq \left\{ \frac{1}{2} \|\tilde{u}_i^T - \tilde{u}_i\|_{L^2(0,T;L^2(\Omega_i))} (2k_i \|\tilde{u}_i^T\|_{L^2(0,T;L^2(\Omega_i))}) \|p_i\|_{L^\infty(0,T;L^2(\Omega_i))} + (1 + a_0) \|\tilde{p}_i\|_{L^2(0,T;L^2(\Omega_i))} \right\} |I_\tau - 1| \|\tilde{u}_i\|_{L^\infty(0,T;L^2(\Omega_i))} \|p_i\|_{L^2(0,T;L^2(\Omega_i))},\]

with a_0 defined as in (4.4). Integrals II_i and IV_i can be estimated in a similar manner and, by employing the uniform boundedness of A_\tau, the Hölder continuity result given in Proposition 6.10 and properties of the mapping F_\tau, we can conclude that

\[\frac{1}{\tau} \sum_{i \in \{+, -\}} \left( I_i + II_i + IV_i \right) \to 0, \text{ as } \tau \to 0.\]

In order to show convergence of the remaining terms to zero, we first rewrite III_i as

\[III_i = \int_0^T \int_{\Omega_i} b_i \delta_i \left\{ \left( |A_\tau \nabla \tilde{u}_i|^q - |A_\tau \nabla \tilde{u}_i|^{q-1} A_\tau \nabla \tilde{u}_i \right) - (|\nabla \tilde{u}_i|^q - |\nabla \tilde{u}_i|^{q-1} \nabla \tilde{u}_i) \right\} \cdot A_\tau \nabla p_i I_\tau \]

\[+ (|\nabla \tilde{u}_i|^q - |\nabla \tilde{u}_i|^{q-1} \nabla \tilde{u}_i) \cdot (I_\tau A_\tau - I) \nabla p_i \right\} \, dx \, ds.\] (6.30)

By employing inequality (3.11) (with \( \eta = 0 \)) and hypothesis (H_2) we can estimate the last line as follows

\[\left| \int_0^T \int_{\Omega_i} b_i \delta_i \left( |A_\tau \nabla \tilde{u}_i|^q - |A_\tau \nabla \tilde{u}_i|^{q-1} \nabla \tilde{u}_i \right) \cdot (I_\tau A_\tau - I) \nabla p_i \right| \, dx \, ds \leq C \|\tilde{u}_i^T - \tilde{u}_i\|_{L^2(0,T;L^2(\Omega_i))} \|\nabla \tilde{u}_i\|_{L^\infty(0,T;L^2(\Omega_i))} \|\nabla A_\tau - I\|_{L^\infty(0,T;L^2(\Omega_i))} \|\nabla p_i\|_{L^2(0,T;L^2(\Omega_i))},\]

which, after division by \( \tau \), tends to 0 as \( \tau \to 0 \), due to Lemma 6.2, Proposition 6.10 and uniform boundedness of \( \|\nabla \tilde{u}_i^T\|_{L^\infty(0,T;L^2(\Omega_i))} \). It remains to estimate the first two lines in (6.30). We will first rewrite them using the representation formula (3.16) as

\[\int_0^T \int_{\Omega_i} b_i \delta_i \left( |A_\tau \nabla \tilde{u}_i|^q - |A_\tau \nabla \tilde{u}_i|^{q-1} A_\tau \nabla \tilde{u}_i \right) - (|\nabla \tilde{u}_i|^q - |\nabla \tilde{u}_i|^{q-1} \nabla \tilde{u}_i) \right\} \cdot A_\tau \nabla p_i I_\tau \, dx \, ds \]

\[= \int_0^T \int_{\Omega_i} b_i \delta_i \left( (A_\tau - I) \nabla (\tilde{u}_i^T - \tilde{u}_i) \right) \int_0^1 |A_\tau \nabla \tilde{u}_i + \sigma A_\tau \nabla (\tilde{u}_i^T - \tilde{u}_i)|^{q-1} \, d\sigma \cdot A_\tau \nabla p_i I_\tau \]

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+ (\nabla (\hat{u}_i^r - \hat{u}_i)) \int_0^1 \left( |A_r \nabla \hat{u}_i + \sigma A_r \nabla (\hat{u}_i^r - \hat{u}_i)\right)^{q-1} - |\nabla \hat{u}_i + \sigma \nabla (\hat{u}_i^r - \hat{u}_i)|^{q-1}) d\sigma \\
+ (q - 1) \int_0^1 \left( \mathcal{L}(A_r \nabla \hat{u}_i + \sigma A_r \nabla (\hat{u}_i^r - \hat{u}_i), A_r \nabla (\hat{u}_i^r - \hat{u}_i)) \\
- \mathcal{L}(\nabla \hat{u}_i + \sigma \nabla (\hat{u}_i^r - \hat{u}_i), \nabla (\hat{u}_i^r - \hat{u}_i)) \right) d\sigma \cdot A_r \nabla p_i \right\} dx ds.

with \( \mathcal{L} \) as in (3.17). For the first line on the right hand side (divided by \( \tau \)) we immediately have convergence to 0, thanks to Lemma 6.2, uniform boundedness of \( A \) and Proposition 6.10. For the remaining terms we further have, due to inequalities (3.11) and (3.18):

\[
\left| \int_0^T \int_{\Omega_i} \nabla (\hat{u}_i^r - \hat{u}_i) \int_0^1 \left( |A_r \nabla \hat{u}_i + \sigma A_r \nabla (\hat{u}_i^r - \hat{u}_i)|^{q-1} \\
- |\nabla \hat{u}_i + \sigma \nabla (\hat{u}_i^r - \hat{u}_i)|^{q-1}) d\sigma \cdot A_r \nabla p_i \right| dx ds \right|
\leq C_q \| \nabla (\hat{u}_i^r - \hat{u}_i) \|_{L^2(0,T;L^2(\Omega_i))} |A_r - I|_{L^\infty(\Omega_i)} \left( \| \nabla \hat{u}_i \|_{L^\infty(0,T;L^\infty(\Omega_i))} \\
+ \| \nabla (\hat{u}_i^r - \hat{u}_i) \|_{L^\infty(0,T;L^\infty(\Omega_i))} \left( (1 + \beta_1^{q-2}) \| \nabla \hat{u}_i \|_{L^\infty(0,T;L^\infty(\Omega_i))} \\
+ (1 + \beta_1^{q-2}) \| \nabla (\hat{u}_i^r - \hat{u}_i) \|_{L^\infty(0,T;L^\infty(\Omega_i))} \right) \right) \alpha_\beta \| \nabla p_i \|_{L^2(0,T;L^2(\Omega_i))}.
\]

Furthermore, by employing estimate (3.19) with \( \eta = 0 \) we obtain:

\[
\left| (q - 1) \int_0^1 \left( \mathcal{L}(A_r \nabla \hat{u}_i + \sigma A_r \nabla (\hat{u}_i^r - \hat{u}_i), A_r \nabla (\hat{u}_i^r - \hat{u}_i)) \\
- \mathcal{L}(\nabla \hat{u}_i + \sigma \nabla (\hat{u}_i^r - \hat{u}_i), \nabla (\hat{u}_i^r - \hat{u}_i)) \right) d\sigma \right|
\leq C_q |A_r - I|_{L^\infty(\Omega_i)} \nabla (\hat{u}_i^r - \hat{u}_i) \left\{ \left( |\nabla \hat{u}_i| + |\nabla (\hat{u}_i^r - \hat{u}_i)| \right)^2 (\beta_1^{q-3} + 1) |\nabla \hat{u}_i|^{q-3} \\
+ |\nabla (\hat{u}_i^r - \hat{u}_i)|^{q-3} (\beta_1^{q-2} + 1 + \beta_1) |\nabla \hat{u}_i| \\
+ |\nabla (\hat{u}_i^r - \hat{u}_i)| \right\} .
\]

valid for \( q > 2 \), from which, by making use of Proposition 6.10, Lemma 6.2 and hypothesis (H1), we finally have \( \frac{1}{T} III_i \to 0 \) as \( \tau \to 0 \) and therefore Proposition 6.12 holds. \( \square \)

For the second property to hold we have to assume that \( p \) is slightly more than \( W^{1,2} \) regular on the subdomains, i.e.:

\[
(H_3) \quad p_{\Omega_i} \in L^{2+\varepsilon}(0,T;W^{1,2+\varepsilon}(\Omega_i)) \text{ for some } \varepsilon > 0.
\]

**Proposition 6.13.** Let \( q > 2 \) and assumptions (5.7) on the coefficients hold. Assume that hypotheses (H1)-(H3) are valid. Then

\[
\lim_{\tau \to 0} \frac{1}{\tau} \left( E(u^r, \Omega^+; E(u, \Omega^+ - E_u(u, \Omega^+)(u^r - u), p) \right)_{\mathcal{X}_*, \mathcal{X}} = 0,
\]

where \( p \) is the adjoint state.
6.5. Auxiliary results

Proof. We have

\[
\langle E(u^T, \Omega_+) - E(u, \Omega_+) - E_u(u, \Omega_+)(u^T - u), p \rangle_{X^*, \hat{X}} \\
= \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_i} -\frac{2k_i}{\lambda_i} ((u_i^T - u_i)(\dot{u}_i^T - \dot{u}_i) + (\dot{u}_i^T - \dot{u}_i)^2) p_i \, dx \, ds
\]

(6.31)

\[ + \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_i} b_i \delta_i \left( |\nabla \dot{u}_i^T|^q - |\nabla \dot{u}_i|^q \right) \cdot \nabla p_i \, dx \, ds. \]

The first sum on the right hand side can be estimated as follows

\[
\left| \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_i} -\frac{2k_i}{\lambda_i} ((u_i^T - u_i)(\dot{u}_i^T - \dot{u}_i) + (\dot{u}_i^T - \dot{u}_i)^2) p_i \, dx \, ds \right|
\]

\[ = \left| \sum_{i \in \{+, -\}} \left[ \int_{\Omega_i} \int_0^T \frac{2k_i}{\lambda_i} (u_i^T - u_i)(\dot{u}_i^T - \dot{u}_i) p_i \, dx \right] \right| + \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_i} \frac{2k_i}{\lambda_i} (u_i^T - u_i)(\dot{u}_i^T - \dot{u}_i) \, p_i \, dx \, ds \]

\[ \leq \sum_{i \in \{+, -\}} \frac{2\delta_i}{\lambda_i} (C_{H^1, L^2})^2 \|u_i^T - u_i\|_{L^\infty(0,T;H^1(\Omega_i))} \|\dot{u}_i^T - \dot{u}_i\|_{L^2(0,T;H^1(\Omega_i))}. \]

since \((u - u^\tau)|_{t=0} = (\dot{u} - \dot{u}^\tau)|_{t=0} = 0 and p|_{t=T} = \dot{p}|_{t=T} = 0. This expression, upon division by \(\tau\), tends to 0 as \(\tau \to 0\), due to Proposition 6.10. The second sum in (6.31) can be rewritten with the help of formula (3.16) as given below

\[
\sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_i} b_i \delta_i \left( |\nabla \dot{u}_i^T|^q - |\nabla \dot{u}_i|^q \right) \cdot \nabla p_i \, dx \, ds
\]

\[ = \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_i} b_i \delta_i \left( |\nabla \dot{u}_i + \sigma(\dot{u}_i^T - \dot{u}_i)|^{q-1} - |\nabla \dot{u}_i|^q \right) \cdot \nabla p_i \, dx \, ds
\]

\[ + \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_i} b_i \delta_i (q - 1) \left( \mathcal{L}(\dot{u}_i + \sigma(\dot{u}_i^T - \dot{u}_i)), \nabla (\dot{u}_i^T - \dot{u}_i) \right) \cdot \nabla p_i \, dx \, ds
\]

\[ := I + II, \]

where \(\mathcal{L}\) can be estimated as in (3.19). By employing inequality (3.11) we obtain

\[
|I| \leq \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_i} b_i \delta_i \left( |\nabla \dot{u}_i^T|^2 - |\nabla \dot{u}_i|^2 \right) \cdot \nabla p_i \, dx \, ds
\]

Making use of Hölder’s inequality and hypothesis \((H_1)\) results in

\[
|I| \leq b\delta C_q \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_i} |\nabla (\dot{u}_i^T - \dot{u}_i)|^2 \cdot |\nabla (\dot{u}_i^T - \dot{u}_i)| \, dx \, ds
\]

\[ + \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_i} |\nabla (\dot{u}_i^T - \dot{u}_i)|^{q-2+\eta} \cdot |\nabla (\dot{u}_i)|^{q-2+\eta} |\nabla p_i| \, dx \, ds \]
Here we can choose \( \eta = \frac{2}{2+\tau} \). Recall that \( \frac{1}{\tau} \| \nabla (\hat{u}^\tau - \hat{u}_i) \|_{L^2(0,T;L^2(\Omega_i))} \) is uniformly bounded for \( |\tau| < \tau_0, \tau \neq 0 \), due to Proposition 6.11 and that thanks to hypothesis \((H_2)\) we have uniform boundedness of \( \| \nabla \hat{u}^\tau \|_{L^\infty(0,T;L^\infty(\Omega_i))} \) as well. This means that we can achieve that \( I \) upon division by \( \tau \) tends to zero as \( \tau \to 0 \).

By employing inequality (3.19), we get the estimate for \( II: \)

\[
|II| \leq \bar{b} \delta C_q \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_i} |\nabla (\hat{u}^\tau - \hat{u}_i)|^{2-\eta} \left\{ |\nabla \hat{u}_i|^{\eta} + |\nabla \hat{u}^\tau_i|^{\eta} \right\} |\nabla p_i| \, dx \, ds
\]

\[
\leq \bar{b} \delta C_q \sum_{i \in \{+, -\}} \left\{ \| \nabla (\hat{u}^\tau - \hat{u}_i) \|^{2-\eta}_{L^2(0,T;L^2(\Omega_i))} \right\} \left\{ \| \nabla \hat{u}_i \|^{\eta}_{L^\infty(0,T;L^\infty(\Omega_i))} + \| \nabla \hat{u}^\tau_i \|^{\eta}_{L^\infty(0,T;L^\infty(\Omega_i))} \right\}
\]

\[
+ \| \nabla \hat{u}^\tau_i \|^{\eta}_{L^\infty(0,T;L^\infty(\Omega_i))} \left\{ \| \nabla \hat{u}_i \|^{\eta}_{L^\infty(0,T;L^\infty(\Omega_i))} + \| \nabla \hat{u}^\tau_i \|^{\eta}_{L^\infty(0,T;L^\infty(\Omega_i))} \right\} \| \nabla p_i \|_{L^\infty(0,T;L^\infty(\Omega_i))},
\]

with \( \eta = \frac{2}{2+\tau} \). Upon division by \( \tau \), due to Propositions 6.10 and 6.11 the right hand side tends to zero as \( \tau \to 0 \).

6.6 Computation of the shape derivative

Let \( u^\tau, u \) satisfy \( \tilde{E}(u^\tau, \tau) = 0 \) and \( E(u, \Omega_+) = 0 \), for \( |\tau| < \tau_0, \tau \in \mathbb{R} \). In that case \( u_\tau = u^\tau \circ F_\tau \) is the solution of \( E(u_\tau, \Omega_{+\tau}) = 0 \). We then have

\[
dJ(u, \Omega_+)h = \lim_{\tau \to 0} \frac{1}{\tau} \int_0^T \int_{\Omega} (j(u^\tau)I_\tau - j(u)) \, dx \, ds
\]

\[
= \int_0^T \int_{\Omega} \left( j'(u) \lim_{\tau \to 0} \frac{u^\tau - u}{\tau} + j(u) \text{div} h \right) \, dx \, ds,
\]

where we have used (similarly to [30 Lemma 2.1]) that

\[
\lim_{\tau \to 0} \frac{1}{\tau} \int_0^T \int_{\Omega} \left( j(u^\tau) - j(u) - j'(u)(u^\tau - u) \right) I_\tau \, dx \, ds
\]

\[
\leq \lim_{\tau \to 0} \frac{1}{\tau} \int_0^T \int_{\Omega} (u^\tau - u)^2 I_\tau \, dx \, ds = 0,
\]

which follows from Proposition 6.10 and the fact that \( I_\tau \) is uniformly bounded for \( \tau \in (-\tau_0, \tau_0) \). By employing the adjoint problem (6.3) and then proceeding as in [30 Theorem 2.1], we obtain

\[
\int_0^T \int_{\Omega} j'(u)(u^\tau - u) \, dx \, ds = \langle E_u(u, \Omega_+)(u^\tau - u), p \rangle_{\tilde{X}^*, \tilde{X}}
\]

\[
= -\langle E(u^\tau, \Omega_+) - E(u, \Omega_+) - E_u(u, \Omega_+)(u^\tau - u), p \rangle_{\tilde{X}^*, \tilde{X}}
\]

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\[-\langle \bar{E}(u^T, \tau) - \bar{E}(u, \tau) - (E(u^T, \Omega) - E(u, \Omega)), \partial \rangle \bar{x}, \bar{x} \]

\[-\langle \bar{E}(u, \tau) - \bar{E}(u, 0), \partial \rangle \bar{x}, \bar{x} \cdot \tau \]

The second and third line divided by \( \tau \) tend to zero, as \( \tau \to 0 \), on account of Propositions 6.12 and 6.13 and we are left with

\[
\lim_{\tau \to 0} \frac{1}{\tau} \int_0^T \int_{\Omega} j'(u)(u^T - u) \, dx \, ds = - \lim_{\tau \to 0} \frac{1}{\tau} \langle \bar{E}(u, \tau) - \bar{E}(u, 0), \partial \rangle \bar{x}, \bar{x}. \]

This limit, representing the (artificial) time derivative of \( \langle \bar{E}(u, \tau), \partial \rangle \bar{x}, \bar{x} \), is typically computed by transforming the expressions \( \bar{E}(u, \tau) \) and \( \partial \) back to \( E(u \circ F_{\tau}^{-1}, \Omega_{+}, \tau) \) and \( p \circ F_{\tau}^{-1} \), and then making use of differentiation rules for \( u \circ F_{\tau}^{-1} \) and \( p \circ F_{\tau}^{-1} \) (see [30] Lemma 2.4 and Examples 1-5). However, these rules assume \( H^2 \) differentiability in space of the primal and the adjoint state, which is too high of a requirement in our case. Instead, we continue with calculating the difference

\[ \langle \bar{E}(u, \tau) - \bar{E}(u, 0), \partial \rangle \bar{x}, \bar{x} = I_+ + I_- \]

where the two terms on the right hand side are given by

\[
I_1 = \int_0^T \int_{\Omega_i} \left\{ \frac{1}{\lambda_i} (1 - 2k_1 u_i) \hat{u}_i \hat{p}_i (I_\tau - 1) + \frac{1}{\theta_i} ((A_\tau - I) \nabla u_i \cdot \nabla p_i + A_\tau \nabla u_i \cdot (A_\tau - I) \nabla p_i + A_\tau \nabla u_i \cdot A_\tau \nabla p_i (I_\tau - 1)) + b_i (1 - \delta_i) ((A_\tau - I) \nabla \hat{u}_i \cdot \nabla p_i + A_\tau \nabla \hat{u}_i \cdot (A_\tau - I) \nabla p_i + A_\tau \nabla \hat{u}_i \cdot A_\tau \nabla p_i (I_\tau - 1)) + b_i \delta_i ((A_\tau - I) \nabla \hat{u}_i - \nabla^{(q-1)} \hat{u}_i)^{q-1} d \sigma \cdot A_\tau \nabla p_i I_\tau \right. \\
+ (q - 1) \int_0^1 |\nabla \hat{u}_i + \sigma (A_\tau - I) \nabla \hat{u}_i|^{q-3} (\nabla \hat{u}_i + \sigma (A_\tau - I) \nabla \hat{u}_i) \cdot (A_\tau - I) \nabla \hat{u}_i \cdot \nabla \hat{u}_i |^{q-1} \nabla \hat{u}_i \cdot ((A_\tau - I) \nabla p_i I_\tau + (I_\tau - 1) \nabla p_i) \left. - \frac{2k_1}{\lambda_i} (u_i)^2 p_i (I_\tau - 1) \right\} dx \, ds ,
\]

\( i \in \{+, -\} \), and we have employed the formula (3.16) to represent the difference \( |A_\tau \nabla \hat{u}_i|^{q-1} A_\tau \nabla \hat{u}_i - |\nabla \hat{u}_i|^{q-1} \nabla \hat{u}_i \). Dividing (6.32) by \( \tau \), passing to the limit and utilizing Lemma 6.2 yields

\[
\lim_{\tau \to 0} \frac{1}{\tau} \langle \bar{E}(u, \tau) - \bar{E}(u, 0), \partial \rangle \bar{x}, \bar{x} \]

\[
= \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_i} \left\{ -\frac{1}{\theta_i} \nabla u_i^T + b_i (1 - \delta_i) \nabla \hat{u}_i^T + b_i \delta_i |\nabla \hat{u}_i|^{q-1} \nabla \hat{u}_i^T (Dh^T \nabla p_i + Dh \nabla p_i) + b_i \delta_i (q - 1) |\nabla \hat{u}_i|^{q-3} (\nabla \hat{u}_i \cdot (Dh^T \nabla p_i)) \right\} \, dx \, ds
\]

(6.33)
Chapter 6. Sensitivity analysis for shape optimization of a focusing lens

\[ + \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_i} \left\{ \frac{1}{\lambda_i} (1 - 2k_i u_i) \dot{u}_i p_i + \frac{1}{\theta_i} \nabla u_i \cdot \nabla p_i + b_i (1 - \delta_i) \nabla \dot{u}_i \cdot \nabla p_i \right. \\
+ b_i \delta_i |\nabla u_i|^{q-1} \nabla \dot{u}_i \cdot \nabla p_i - \frac{2k_i}{\lambda_i} (\dot{u}_i)^2 p_i \right\} \text{div} h \, dx \, ds, \]

and we can now express the Eulerian derivative:

**Theorem 6.14.** (Weak shape derivative) Let \( q > 2 \), \( u_0, u_1 \in W_0^{1,q+1}(\Omega) \), and assumptions \([5,7]\) on coefficients hold. Assume that \((H_1)-(H_3)\) are valid. Then the shape derivative of \( J \) at \( \Omega_+ \) with respect to \( h \in C^{1,1}(\overline{\Omega}, \mathbb{R}^d) \) can be represented as

\[
dJ(u, \Omega_+) h = \int_0^T \int_{\Omega} \left\{ \frac{1}{\theta} \nabla u \cdot \nabla p + b(1 - \delta) \nabla \dot{u}^T (Dh \nabla p) + b\delta |\nabla u|^{q-1} \nabla \dot{u} \cdot \nabla p \right\} \text{div} h \, dx \, ds \\
- \int_0^T \int_{\Omega} \left\{ \frac{1}{\lambda} (1 - 2k u) \dot{u} \cdot \nabla p + \frac{1}{\theta} \nabla u \cdot \nabla p + b(1 - \delta) \nabla \dot{u} \cdot \nabla p \right. \\
+ b\delta |\nabla u|^{q-1} \nabla \dot{u} \cdot \nabla p - \frac{2k}{\lambda} (\dot{u})^2 p - j(u) \right\} \text{div} h \, dx \, ds. \tag{6.34} \]

Note that the integrals in (6.34) are well-defined thanks to hypothesis \((H_1)\), and for them to be well-defined hypothesis \((H_3)\) is actually not necessary.

Theorem 6.14 gives us the shape derivative of the cost functional in terms of the volume integrals, which is in [8] regarded as a weak shape derivative. However, an obvious advantage of the volume expression of the shape derivative is that it allows for a lower regularity of shapes as well as lower regularity of the primal and the adjoint state. Recently there have been suggestions that the domain representation is also advantageous in terms of easiness of computation and numerical implementations (see, for example, \([27, 43]\)), especially in the case of transmission problems where shape derivatives given in terms of the boundary integrals contain jumps of functions over the interfaces, which is numerically a delicate task to perform. We also refer to [57] for an up-to-date inquiry, where a novel Steklov-Poincaré type intrinsic metric in shape spaces was developed which enables to work with both domain based and boundary based shape derivative expressions.

### 6.6.1 Strong shape derivative.

In order to express the shape derivative in the form required by Theorem 5.2 we would have to apply Green’s theorem to the last two lines in (6.33), which is not allowed since \( u \) and \( p \) are not sufficiently regular. However, we know thanks to the results from Chapter 5 that if the domains are sufficiently smooth and \( \nabla \dot{u} \) is bounded in \( L^\infty(0, T; L^\infty(\Omega)) \), the state variable exhibits \( H^2 \)-regularity on each of the subdomains. This result together with an assumption regarding the regularity of the trace of \( \nabla u_{\Omega_i} \) and \( \nabla p_{\Omega_i} \) on \( \Gamma \) makes expressing the shape derivative of the cost functional in terms of the boundary integrals possible. Let \( \partial \Omega \) and \( \Gamma = \partial \Omega_+ \) be \( C^{1,1} \) regular. Due to Theorem 5.7 we know that \( u_i \in H^1(0, T; H^2(\Omega_i)), i \in \{+, -\} \).

To be able to express the shape derivative in terms of the boundary integrals over \( \Gamma \), we first impose additional regularity hypotheses on \( u \) and \( p \):

**\((H_4)\)** \( \text{tr}_H^{\Omega_i} \nabla p \in L^1(0, T; L^1(\Gamma)), i \in \{+, -\}, \)
where $\frac{\partial}{\partial \Gamma} \nabla u$ and $\frac{\partial}{\partial \Gamma} \nabla p$ stand for the trace of $\nabla u|_{\Omega}$ and $\nabla p|_{\Omega}$, respectively, on $\Gamma$. Hypotheses $(\mathcal{H}_4)$ and $(\mathcal{H}_5)$ will ensure that the forthcoming boundary integrals are well-defined. Note that they do not follow from the previous hypotheses $(\mathcal{H}_1)$-$$(\mathcal{H}_3)$ and regularity results, partially due to the lack of an appropriate trace theorem in the limiting $L^\infty$-case (see Theorem 3.6 and 3.7).

Next, we introduce sufficiently smooth in space approximations of the adjoint state in $H^1(0, T; H^1(\Omega_i))$. Fix $i \in \{+, -\}$. Let $\{p_{i,m}\}_{m=1}^\infty \subset H^1(0, T; C^\infty(\Omega_i))$ be a sequence that converges to $p_i$ in $H^1(0, T; H^1(\Omega_i))$ and such that $p_{i,m} = p_i$ on $\partial \Omega_i$ (cf. Theorem 3.9).

We can then approximate $(6.33)$ as

$$
\lim_{\tau \to 0} \frac{1}{\tau} \langle \tilde{E}(u, \tau) - \tilde{E}(u, 0), p \rangle_{\hat{X}^*, \hat{X}} = \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_i} \left\{ -\frac{1}{\tilde{\varrho}_i} \nabla u_i^T + b_i(1 - \delta_i) \nabla \hat{u}_i^T + b_i \delta_i |\nabla \hat{u}_i|^{q-1} \nabla \hat{u}_i^T \right\} \left( D\hat{H}^T \nabla p_{i,m} \right) + D\hat{h} \nabla p_{i,m} + b_i \delta_i (q - 1) |\nabla \hat{u}_i|^{q-3} (\nabla \hat{u}_i \cdot (- D\hat{h})^T \nabla \hat{u}_i) \left( \nabla \hat{u}_i \cdot \nabla (p_i - p_{i,m}) \right) \right) dx \, ds
$$

(6.35)

$$
+ \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_i} \left\{ \frac{1}{\nu_i} \nabla \hat{u}_i \cdot \nabla p_{i,m} + b_i \delta_i (1 - \delta_i) \nabla \hat{u}_i \cdot \nabla p_{i,m} \right\} dx \, ds + R_1(p_i, p_{i,m}),
$$

where the error term is given by

$$
R_1(p_i, p_{i,m}) = \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_i} \left\{ \left( \frac{1}{\nu_i} \nabla \hat{u}_i \right)^2 \left( \nabla \hat{u}_i \right)^2 \nabla p_{i,m} \right\} + b_i \delta_i (1 - \delta_i) \nabla \hat{u}_i \cdot \nabla (p_i - p_{i,m})
$$

Since $u_i$ and $p_{i,m}$ are sufficiently smooth, we are allowed to employ Green’s theorem in (6.35). This will cause the terms containing $D\hat{h}$ (not included in $R_1$) to cancel out, and we arrive at

$$
\lim_{\tau \to 0} \frac{1}{\tau} \langle \tilde{E}(u, \tau) - \tilde{E}(u, 0), p \rangle_{\hat{X}^*, \hat{X}} = \sum_{i \in \{+, -\}} \int_0^T \int_{\partial \Omega_i} \left\{ \frac{1}{\nu_i} (1 - 2k_i u_i) \nabla \hat{u}_i \cdot \nabla p_{i,m} + \frac{1}{\nu_i} \nabla \hat{u}_i \cdot \nabla p_{i,m} \right\} \cdot h \nabla u_i dx \, ds
$$

$$
+ b_i (1 - \delta_i) \nabla \hat{u}_i \cdot \nabla p_{i,m} + b_i \delta_i |\nabla \hat{u}_i|^{q-1} \nabla \hat{u}_i \cdot \nabla p_{i,m} - \frac{2k_i}{\nu_i} (\hat{u}_i)^2 \nabla p_{i,m} dx \, ds
$$

- \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_i} \left\{ \frac{1}{\nu_i} (1 - 2k_i u_i) \nabla \hat{p}_{i,m} \cdot h + \frac{1}{\nu_i} \nabla u_i \cdot \nabla (\nabla \hat{p}_{i,m} \cdot h) \right\} dx \, ds.
\[ + b_i(1 - \delta_i)\nabla \dot{u}_i \cdot \nabla (\nabla p_{i,m}^T h) + b_i \delta_i |\nabla \dot{u}_i|^{q-1} \nabla \dot{u}_i \cdot \nabla (\nabla p_{i,m}^T h)\] 
\[ - \sum_{i \in \{+,-\}} \int_0^T \int_{\Omega_i} \left\{ \frac{1}{\lambda_i} (1 - 2k_i u_i) \tilde{p}_{i,m,i} (\nabla u_i^T h) + \frac{1}{\theta_i} \nabla p_{i,m} \cdot \nabla (\nabla u_i^T h) \right\} dx \, ds \]
\[ - b_i(1 - \delta_i) \nabla \tilde{p}_{i,m} \cdot \nabla (\nabla u_i^T h) - b_i \delta_i (G_{u_i}(\nabla p_{i,m}))' \cdot \nabla (\nabla u_i^T h)\] 
\[ + R_1(p_i, p_{i,m}) + R_2(p_i, p_{i,m}) + R_3(p_i, p_{i,m}),\]

where the approximation error terms are given by

\[ R_2(p_i, p_{i,m}) = \sum_{i \in \{+,-\}} \int_0^T \int_{\Omega_i} \left\{ \frac{1}{\lambda_i} (1 - 2k_i u_i) (\tilde{p}_i - \tilde{p}_{i,m,i}) (\nabla u_i^T h) \right\} dx \, ds, \]

\[ R_3(p_i, p_{i,m}) = - \sum_{i \in \{+,-\}} \int_0^T \int_{\partial \Omega_i} \left\{ \frac{1}{\lambda_i} (1 - 2k_i u_i) \tilde{u}_i (p_i - p_{i,m}) + \frac{1}{\theta_i} \nabla u_i \cdot \nabla (p_i - p_{i,m}) \right\} hT n_i dx \, ds, \]

Remark 6.15. If \( u \in W^{1,\infty}(0, T; W^{1,\infty}(\Omega)) \), we are allowed to use \( \phi \in L^2(0, T; H^1_0(\Omega)) \) as a test function in (4.61). To see this, let \( \phi_j \in L^2(0, T; C^\infty_0(\Omega)), \phi_j \to \phi \) in \( L^2(0, T; H^1_0(\Omega)) \) as \( j \to \infty \). Then

\[ \int_0^T \int_{\Omega} \frac{1}{\lambda} (1 - 2k u) \tilde{u} \phi \, d\tau \, d\Omega + \frac{1}{\theta} \nabla u \cdot \nabla \phi + b(1 - \delta) \nabla u \cdot \nabla \phi \]
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\[ + b\delta |\nabla \hat{u}|^{q-1} \nabla \hat{u} \cdot \nabla \phi - \frac{2k}{\lambda} (\hat{u})^2 \phi \} \, dx \, ds \]

\[ = \int_0^T \int_{\Omega} \left\{ \frac{1}{\lambda} (1 - 2ku)\hat{u}(\phi - \phi_j) + \frac{1}{\varrho} \nabla u \cdot \nabla (\phi - \phi_j) + b(1 - \delta)\nabla \hat{u} \cdot \nabla (\phi - \phi_j) \right\} \, dx \, ds \]

\[ + b\delta |\nabla \hat{u}|^{q-1} \nabla \hat{u} \cdot \nabla (\phi - \phi_j) - \frac{2k}{\lambda} (\hat{u})^2 (\phi - \phi_j) \} \, dx \, ds \]

\[ \leq \frac{1}{\Delta} \left( (1 + a_0) \| \hat{u} \|_{L^2(0,T;L^2(\Omega))} + 2\bar{K}(C_{H^1_0,L^1})^2 \sqrt{T} \| u \|_{L^\infty(0,T;H^1_0(\Omega))} \| \phi - \phi_j \|_{L^2(0,T;L^2(\Omega))} \right) \]

\[ + \frac{1}{\varrho} \| \nabla u \|_{L^2(0,T;L^2(\Omega))} + \bar{b}(1 - \delta) \| \nabla \hat{u} \|_{L^2(0,T;L^2(\Omega))} \]

\[ + \bar{b}\delta \| \nabla \hat{u} \|_{L^\infty(0,T;L^2(\Omega))} \| \nabla \hat{u} \|_{L^2(0,T;L^2(\Omega))} \| \nabla (\phi - \phi_j) \|_{L^2(0,T;L^2(\Omega))} \]

\[ \to 0, \quad \text{as} \quad j \to \infty. \]

Note that \( \nabla u_T^T h \in L^2(0, T; H^1(\Omega_t)) \) and \( \nabla p_{T,m}^T h \in L^2(0, T; H^1(\Omega_t)) \), however functions

\[ \tilde{\phi}(x, t) := \begin{cases} \nabla u_T^T(x, t) h(x) & x \in \Omega_+ \\ \nabla u_T^T(x, t) h(x) & x \in \Omega_- \end{cases} \]

\[ \tilde{\zeta}(x, t) := \begin{cases} \nabla p_{T,m}^T(x, t) h(x) & x \in \Omega_+ \\ \nabla p_{T,m}^T(x, t) h(x) & x \in \Omega_- \end{cases} \]

do not have to be continuous across the boundary \( \Gamma \) and we cannot use them directly as test functions in the weak formulations of the state and the adjoint problem. We can instead employ the two-domain weak formulations which results in

\[ \lim_{\tau \to 0} \frac{1}{\tau} \langle \tilde{E}(u, \tau) - \tilde{E}(u, 0), p \rangle_{\tilde{X}^*, \tilde{X}} \]

\[ = \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_{i,m}} \left\{ \frac{1}{\lambda_i} (1 - 2kiu_i)\hat{u}_i p_i + \frac{1}{\varrho_i} \nabla u_i \cdot \nabla p_i \right\} \, dx \, ds \]

\[ + b_i (1 - \delta_i) \nabla \hat{u}_i \cdot \nabla p_i + b_i \delta_i |\nabla \hat{u}_i|^{q-1} |\nabla \hat{u}_i| \cdot \nabla p_i - \frac{2k_i}{\lambda_i} (\hat{u}_i)^2 p_i \} \, h^T n_i \, dx \, ds \]

\[ - \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_{i,m}} \left\{ \frac{1}{\varrho_i} \frac{\partial u_i}{\partial n_i} + b_i (1 - \delta_i) \frac{\partial \hat{u}_i}{\partial n_i} + b_i \delta_i |\nabla \hat{u}_i|^{q-1} \frac{\partial \hat{u}_i}{\partial n_i} \right\} (\nabla p_{T,m}^T h) \, dx \, ds \]

\[ - \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_{i,m}} \left\{ \frac{1}{\varrho_i} \frac{\partial p_i}{\partial n_i} - b_i (1 - \delta_i) \frac{\partial \hat{p}_i}{\partial n_i} - b_i \delta_i (G_{u_i}(\nabla p_i) \cdot n_i) \right\} (\nabla u_T^T h) \, dx \, ds \]

\[ - \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_{i,m}} f_i' (u_i) (\nabla u_T^T h) \, dx \, ds + R_1(p_i, p_{i,m}) + R_2(p_i, p_{i,m}) + R_3(p_i, p_{i,m}). \]

Finally, this can be rewritten as

\[ \lim_{\tau \to 0} \frac{1}{\tau} \langle \tilde{E}(u, \tau) - \tilde{E}(u, 0), p \rangle_{\tilde{X}^*, \tilde{X}} \]

\[ = \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_{i,m}} \left\{ \frac{1}{\lambda_i} (1 - 2kiu_i)\hat{u}_i p_i + \frac{1}{\varrho_i} \nabla u_i \cdot \nabla p_i \right\} \, dx \, ds \]

\[ + b_i (1 - \delta_i) \nabla \hat{u}_i \cdot \nabla p_i + b_i \delta_i |\nabla \hat{u}_i|^{q-1} |\nabla \hat{u}_i| \cdot \nabla p_i - \frac{2k_i}{\lambda_i} (\hat{u}_i)^2 p_i \} \, h^T n_i \, dx \, ds \]

\[ - \sum_{i \in \{+, -\}} \int_0^T \int_{\Omega_{i,m}} \left\{ \frac{1}{\varrho_i} \frac{\partial u_i}{\partial n_i} + b_i (1 - \delta_i) \frac{\partial \hat{u}_i}{\partial n_i} + b_i \delta_i |\nabla \hat{u}_i|^{q-1} \frac{\partial \hat{u}_i}{\partial n_i} \right\} (\nabla p_{T,m}^T h) \, dx \, ds \]
\[ - \sum_{i \in \{+,-\}} \int_{0}^{T} \int_{\partial \Omega_{i}} \left\{ \frac{1}{\varrho_{i}} \frac{\partial p_{i}}{\partial n_{i}} - b_{i}(1 - \delta_{i}) \frac{\partial \dot{u}_{i}}{\partial n_{i}} - b_{i}\delta_{i} \left( G_{u_{i}}(\nabla p_{i}) \cdot n_{i} \right) \right\} (\nabla u_{i}^{T} h) \, dx \, ds \\
- \sum_{i \in \{+,-\}} \int_{0}^{T} \int_{\Omega_{i}} \frac{1}{\varrho_{i}} \frac{\partial}{\partial n_{i}} \left( \nabla u_{i}^{T} h \right) \, dx \\
+ R_{1}(p_{i}, p_{i,m}) + R_{2}(p_{i}, p_{i,m}) + R_{3}(p_{i}, p_{i,m}) + R_{4}(p_{i}, p_{i,m}), \]

(the second sum is written in terms of \( p_{i} \) plus the error \( R_{4} \)) with

\[ R_{4}(p_{i}, p_{i,m}) = \sum_{i \in \{+,-\}} \int_{0}^{T} \int_{\partial \Omega_{i}} \left\{ \frac{1}{\varrho_{i}} \frac{\partial u_{i}}{\partial n_{i}} + b_{i}(1 - \delta_{i}) \frac{\partial \dot{u}_{i}}{\partial n_{i}} + b_{i}\delta_{i} |\nabla \dot{u}_{i}|^{q-1} \frac{\partial u_{i}}{\partial n_{i}} \right\} \nabla (p_{i} - p_{i,m})^{T} h \, dx \, ds. \]

Next, we want to show that

\[ R(p_{i}, p_{i,m}) := R_{1}(p_{i}, p_{i,m}) + R_{2}(p_{i}, p_{i,m}) + R_{3}(p_{i}, p_{i,m}) + R_{4}(p_{i}, p_{i,m}) \]

tends to zero as \( m \to \infty \). We will focus here on the estimates for the boundary integrals.

Since \( p_{i} - p_{i,m} = 0 \) on \( \Gamma \), we know that \( \nabla_{\Gamma}(p_{i} - p_{i,m}) = 0 \), where \( \nabla_{\Gamma} \) denotes the tangential gradient. This further implies that

\begin{align}
(6.36) & \quad \nabla (p_{i} - p_{i,m}) |_{\Gamma} \cdot h = \frac{\partial (p_{i} - p_{i,m})}{\partial n_{i}} (h \cdot n_{i}), \\
(6.37) & \quad \nabla (p_{i} - p_{i,m}) |_{\Gamma} \cdot \nabla u_{i} |_{\Gamma} = \frac{\partial}{\partial n_{i}} (p_{i} - p_{i,m}) \frac{\partial u_{i}}{\partial n_{i}}.
\end{align}

Due to (6.36) and the fact that \( h = 0 \) on \( \partial \Omega_{-} \setminus \Gamma \), we can estimate

\[ R_{4}(p_{i}, p_{i,m}) = \sum_{i \in \{+,-\}} \int_{0}^{T} \int_{\partial \Omega_{i}} \left\{ \frac{1}{\varrho_{i}} \frac{\partial u_{i}}{\partial n_{i}} + b_{i}(1 - \delta_{i}) \frac{\partial \dot{u}_{i}}{\partial n_{i}} \\
+ b_{i}\delta_{i} |\nabla \dot{u}_{i}|^{q-1} \frac{\partial u_{i}}{\partial n_{i}} \right\} \frac{\partial (p_{i} - p_{i,m})}{\partial n_{i}} h^{T} n_{i} \, dx \, ds \\
\leq |h|_{L^{\infty}(\Gamma)} \sum_{i \in \{+,-\}} \left( \frac{1}{\varrho_{i}} \left\| \frac{\partial u_{i}}{\partial n_{i}} \right\|_{L^{2}(0,T;H^{1/2}(\Gamma))} + b_{i}(1 - \delta_{i}) \left\| \frac{\partial \dot{u}_{i}}{\partial n_{i}} \right\|_{L^{2}(0,T;H^{1/2}(\Gamma))} \\
+ b_{i}\delta_{i} \left\| \nabla \dot{u}_{i} \right\|_{L^{\infty}(0,T;L^{\infty}(\Gamma))}^{q-1} \left\| \frac{\partial u_{i}}{\partial n_{i}} \right\|_{L^{2}(0,T;H^{1/2}(\Gamma))} \right) \\
\leq C |h|_{L^{\infty}(\Gamma)} \sum_{i \in \{+,-\}} \left( \frac{1}{\varrho_{i}} \left\| \frac{\partial u_{i}}{\partial n_{i}} \right\|_{L^{2}(0,T;H^{1/2}(\Gamma))} + b_{i}(1 - \delta_{i}) \left\| \frac{\partial \dot{u}_{i}}{\partial n_{i}} \right\|_{L^{2}(0,T;H^{1/2}(\Gamma))} \\
+ b_{i}\delta_{i} \left\| \nabla \dot{u}_{i} \right\|_{L^{\infty}(0,T;L^{\infty}(\Gamma))} \left\| \frac{\partial u_{i}}{\partial n_{i}} \right\|_{L^{2}(0,T;H^{1/2}(\Omega_{i}))} \right) \|p_{i} - p_{i,m}\|_{L^{2}(0,T;H^{1}(\Omega))} \\
\rightarrow 0, \quad \text{as } m \to \infty.\]

Here we have made use of the fact that since \( u_{i} \in H^{1}(0,T;H^{2}(\Omega_{i})) \), we have \( \frac{\partial u_{i}}{\partial n_{i}} \in H^{1}(0,T;H^{1/2}(\partial \Omega_{i})) \) (see Theorem 3.8), provided that \( \Omega_{i} \) has a \( C^{1,1} \) boundary, which we have assumed. Similarly, by employing (6.37), we obtain

\[ R_{3}(p_{i}, p_{i,m}) = - \sum_{i \in \{+,-\}} \int_{0}^{T} \int_{\partial \Omega_{i}} \left\{ \frac{1}{\lambda_{i}} (1 - 2k_{i}u_{i}) \dot{u}_{i}(p_{i} - p_{i,m}) + \frac{1}{\varrho_{i}} \frac{\partial}{\partial n_{i}} \frac{\partial (p_{i} - p_{i,m})}{\partial n_{i}} \right\} \]
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\[ + b_i (1 - \delta_i) \frac{\partial u_i}{\partial n_i} \frac{\partial (p_i - p_{i,m})}{\partial n_i} + b_i \delta_i |\nabla u_i|^{q-1} \frac{\partial u_i}{\partial n_i} \frac{\partial (p_i - p_{i,m})}{\partial n_i} \]
\[ - \frac{2k_i}{\lambda_i} (\ddot{u}_i)^2 (p_i - p_{i,m}) \right\} h_T n_i \, dx \, ds \]
\[ = \sum_{i \in \{+,-\}} \int_0^T \int_{\partial \Omega_i} \left\{ \frac{1}{\lambda_i} (1 - 2k_i u_i) \dot{u}_i p_i + \frac{1}{\varrho_i} \nabla u_i \cdot \nabla p_i \right. \]
\[ + b_i (1 - \delta_i) \nabla \dot{u}_i \cdot \nabla p_i + b_i \delta_i |\nabla \dot{u}_i|^{q-1} \nabla \dot{u}_i \cdot \nabla p_i - \frac{2k_i}{\lambda_i} (\ddot{u}_i)^2 p_i \right\} h_T n_i \, dx \, ds \]
\[ - \sum_{i \in \{+,-\}} \int_0^T \int_{\partial \Omega_i} \left\{ \frac{1}{\lambda_i} \frac{\partial u_i}{\partial n_i} + b_i (1 - \delta_i) \frac{\partial \dot{u}_i}{\partial n_i} + b_i \delta_i |\nabla \dot{u}_i|^{q-1} \frac{\partial \dot{u}_i}{\partial n_i} \right\} (\nabla \dot{p}_i h_T) \, dx \, ds \]
\[ - \sum_{i \in \{+,-\}} \int_0^T \int_{\partial \Omega_i} \left\{ \frac{1}{\varrho_i} \frac{\partial p_i}{\partial n_i} - b_i (1 - \delta_i) \frac{\partial \dot{p}_i}{\partial n_i} - b_i \delta_i (G_{u_i} (\nabla p_i) \cdot n_i)' \right\} (\nabla u_i^T h) \, dx \, ds \]
\[ - \sum_{i \in \{+,-\}} \int_0^T \int_{\Omega_i} j'(u_i) (\nabla u_i^T h) \, dx \, ds. \]

Assume that \( u_d \in L^2(0, T; H^1_0(\Omega)) \). We can utilize the Stokes theorem and the fact that \( h = 0 \) on \( \partial \Omega_0 \) to acquire

\[ \sum_{i \in \{+,-\}} \int_0^T \int_{\Omega_i} j(u_i) \, \text{div} \, h \, dx \, ds + \sum_{i \in \{+,-\}} \int_0^T \int_{\Omega_i} j'(u_i) (\nabla u_i^T h) \, dx \, ds \]
\[ = \sum_{i \in \{+,-\}} \int_0^T \int_{\Gamma} j(u_i) h^T n_i \, dx \, ds, \]

which leads to the shape derivative given in terms of the boundary integrals

\[ dJ(u, \Omega_+)h = - \int_0^T \int_{\Gamma} \left[ \frac{1}{\lambda} (1 - 2k u) \dot{u} p + \frac{1}{\varrho} \nabla u \cdot \nabla p + b(1 - \delta) \nabla \dot{u} \cdot \nabla p \right. \]
\[ + b \delta |\nabla \dot{u}|^{q-1} \nabla \dot{u} \cdot \nabla p - \frac{2k}{\lambda} (\ddot{u})^2 p \right\} h^T n_+ \, dx \, ds \]
\[ + \int_0^T \int_{\Gamma} \left\{ \frac{1}{\varrho} \nabla u + b(1 - \delta) \nabla \dot{u} + b \delta |\nabla \dot{u}|^{q-1} \nabla \dot{u} \right\} \cdot n_+ (\nabla p \cdot h) \right] \, dx \, ds \]
Theorem 6.16

Here we have made use of the fact that $\left\{(u - u_d)^2\right\} = 0$. The expression for the shape derivative can be slightly simplified. For the second and third integral on the right hand side, by employing the fact that $(\Omega, \mathbb{R}^d)$, we obtain the following identities

$$\text{[x] = [y] = 0} \implies \text{[xy] = 0},$$

we obtain the following identities

$$\left\lfloor \left( \frac{1}{\varrho} \nabla u + b(1 - \delta) \nabla \dot{u} + b \delta \nabla \dot{u} \right) \mathbf{v} \cdot n_+ \right\rfloor$$

$$= \left\lfloor \left( \frac{1}{\varrho} \frac{\partial u}{\partial n_+} + \frac{\partial p}{\partial n_+} + b(1 - \delta) \frac{\partial \dot{u}}{\partial n_+} + b \delta \nabla \dot{u} \right) \mathbf{v} \cdot n_+ \right\rfloor$$

and similarly

$$\left\lfloor \left( \frac{1}{\varrho} \nabla p - b(1 - \delta) \nabla \ddot{p} - b \delta (G_u(\nabla p)) \right) \mathbf{v} \cdot n_+ \right\rfloor$$

Here we have made use of the fact that $\text{[\nabla u] = [\nabla p] = 0}$. We finally obtain

**Theorem 6.16.** (Strong shape derivative) Let $\partial \Omega$ and $\Gamma = \partial \Omega_+$ be $C^{1,1}$, $u_0|\Omega \in H^2(\Omega)$, $u_0, u_1 \in W_0^{1,q+1}(\Omega)$, $q > 2$, and let assumptions (5-7) on the coefficients in the state equation and hypotheses (H1)-(H5) hold true. Assume that $u_d \in L^2(0, T; H^1_0(\Omega))$. The shape derivative of $J$ at $\Omega_+$ in the direction of a vector field $h \in C^{1,1}(\Omega, \mathbb{R}^d)$ is given by

$$dJ(u, \Omega_+)h = \int_0^T \int_\Gamma \left[ -\frac{1}{\lambda} (1 - 2ku) \ddot{u}p - \frac{1}{\varrho} \nabla u \cdot \nabla p \right.$$

$$- b((1 - \delta) \nabla \dot{u} - 1) \nabla \ddot{u} \cdot \nabla p + \frac{2k}{\lambda} (\dot{u})^2 p$$

$$+ \frac{2}{\varrho} \frac{\partial u}{\partial n_+} \frac{\partial p}{\partial n_+} + 2b((1 - \delta) \nabla \dot{u} - 1) \frac{\partial u}{\partial n_+} \frac{\partial p}{\partial n_+}$$

$$+ b \delta (q - 1) \nabla \dot{u} \nabla \dot{u} \cdot \nabla p \right] h \cdot \frac{\partial \dot{u}}{\partial n_+} \right\rfloor d\Gamma n_+ \cdot dx \, ds.$$

(6.8)

The boundary integrals in (6.38) are well-defined thanks to hypotheses (H4) and (H5).
Conclusion and outlook

This thesis was devoted to contributing to mathematical investigations of nonlinear acoustics in terms of well-posedness, regularity results, considerations of coupling over a common interface, and shape optimization. The following results were obtained:

(i) Westervelt’s equation with strong nonlinear damping of $q$-Laplace type was considered under practically relevant Neumann and absorbing boundary conditions. We showed local in time well-posedness with sufficiently small initial and boundary data. Moreover, we investigated the coupled problem motivated by lithotripsy, and obtained the well-posedness result as well.

(ii) We obtained higher interior regularity results for the model in question as well as for the coupled system. We also showed that this higher regularity can be extended to the boundary of the subdomains if the gradient of the acoustic pressure remains essentially bounded in space and time - a result of interest in numerical calculations, and shape optimization problems governed by this system.

(iii) Finally, we studied a shape optimization problem motivated by the efficient treatment of kidney stones in lithotripsy. We have computed, through a variational approach and in a mathematically rigorous way, the weak and the strong shape derivative of the cost functional determining the acoustic pressure of high intensity ultrasound when focusing is performed by an acoustic lens. The results obtained here represent a sound basis for constructing the optimally focusing lens and improving the overall performance of lithotripters.

There are several possible research directions in relation to this thesis which could be taken in the future. As a continuation of research presented in Chapter 6, developing and implementing a suitable optimization algorithm based on the obtained shape derivative is a natural next step. Implementation of such an algorithm, however, contains another highly nontrivial objective, namely developing and implementing an efficient solver for the state and the adjoint problem.

It would also be of interest, having applications in mind, to include additional constraints in the optimization problem, for instance appropriate geometric constraints on the acoustic lens.

Considerations of a physically more involved elastic model of the focusing lens and an elastic-acoustic coupling (2.11) under Neumann and absorbing boundary conditions are of interest also, as well as the shape optimization problem governed by this model.
Conclusion and outlook
Notational convention

Physical quantities

\(u\) acoustic pressure
\(v\) acoustic particle velocity
\(c\) speed of sound
\(\varrho\) mass density
\(\psi\) acoustic velocity potential, \(\varrho \dot{\psi} = u\)
\(\lambda\) bulk modulus
\(B/A\) parameter of nonlinearity
\(b\) diffusivity of sound in the Westervelt equation with constant coefficients, see page 8 or the quotient of the diffusivity of sound and the bulk modulus in the coupled model, see page 38
\(k\) 
\[= \frac{1}{\lambda} \left(1 + \frac{B}{2A}\right)\]

Sets

\(\Omega\) domain in \(\mathbb{R}^d, d \in \{1, 2, 3\}\)
\(\overline{\Omega}\) the closure of \(\Omega\)
\(\Omega_+\) the lens subdomain, see page 45
\(\Omega_-\) the fluid subdomain, see page 45

Function spaces

\(C^{l,\alpha}(\overline{\Omega})\) space of \(l\)-times continuously differentiable functions whose \(l\)-th partial derivatives are bounded and Hölder continuous with exponent \(\alpha\)
\(C^{0,1}(\overline{\Omega})\) space of Lipschitz continuous functions
\(L^r(\Omega)\) space of measurable functions that are \(r\)-integrable,
\[ |u|_{L^r(\Omega)} := \left( \int_{\Omega} |u|^r \, dx \right)^{1/r}, \quad 1 \leq r < \infty \]

\( L^\infty(\Omega) \) space of essentially bounded functions
\[ |u|_{L^\infty(\Omega)} := \text{ess sup}_{x \in \Omega} |u(x)| \]

\( W^{l,r}(\Omega) \) space of all locally summable functions \( u : \Omega \to \mathbb{R} \) such that for each \( \alpha \) with \( |\alpha| \leq l, \ l \in \mathbb{N}_0 \), \( D^\alpha u \) exists and belongs to \( L^r(\Omega) \),
\[ |u|_{W^{l,r}(\Omega)} := \left( \sum_{|\alpha| \leq l} \int_{\Omega} |D^\alpha u|^r \, dx \right)^{1/r}, \quad 1 \leq r < \infty, \]
\[ |u|_{W^{l,\infty}(\Omega)} := \sum_{|\alpha| \leq l} \text{ess sup}_{\Omega} |D^\alpha u| \]

\( C(0, T; V) \) space of continuous functions \( u : [0, T] \to V \) with
\[ \|u\|_{C(0, T; V)} = \max_{t \in [0, T]} |u(t)|_V < \infty \]

\( C^k(0, T; V) \) space of \( k \) time continuously differentiable functions
\[ u : [0, T] \to V \text{ with } \|u\|_{C^k(0, T; V)} = \sum_{j=0}^k \|d^j u/dt^j\|_{C(0, T; V)} < \infty \]

\( L^r(0, T; V) \) space of measurable functions \( u : [0, T] \to V \) with
\[ \|u\|_{L^r(0, T; V)} = \left( \int_0^T |u(t)|^r_V \, dt \right)^{1/r}, \quad 1 \leq r < \infty, \]
\[ \|u\|_{L^\infty(0, T; V)} = \text{ess sup}_{[0, T]} |u(t)|_V < \infty \]

\( W^{1,r}(0, T; V) \) space of measurable functions \( u \in L^r(0, T; V) \) whose weak derivative \( d^j u/dt^j \in L^r(0, T; V) \), \( 0 \leq j \leq l \)
\[ \|u\|_{W^{1,r}(0, T; V)} := \left( \sum_{j=0}^k \|d^j u/dt^j\|_{L^r(0, T; V)}^r \right)^{1/r}, \quad 1 \leq r < \infty, \]
\[ \|u\|_{W^{1,\infty}(0, T; V)} := \sum_{j=0}^k \|d^j u/dt^j\|_{L^\infty(0, T; V)} \]

**Operators**

\( \dot{\cdot} \) first derivative with respect to the time variable
\( \ddot{\cdot} \) second derivative with respect to the time variable
\( D^l_r \) \( r \)-th difference quotient of size \( l \), see page 41
\( \text{tr}_\Gamma \) trace operator, see page 14
\( \nabla \) spatial gradient: \( \nabla u := (\partial u/\partial x_1, \ldots, \partial u/\partial x_d) \)
\( \nabla_\Gamma \) tangential gradient on \( \Gamma \)
\( \text{div}, \nabla \cdot \) divergence with respect to the spatial variables
\[ \Delta \quad \text{Laplace operator: } \Delta u := \text{div}(\nabla u) \]

\[ \Delta_q \quad q\text{-Laplace operator: } \Delta_q u := \text{div}|\nabla u|^{q-1} \nabla u, \; q \geq 1 \]

**Constants**

- \(C, C\) generic positive constants
- \(C_q\) positive constant depending only on \(q\)
- \(C_1^{tr}\) norm of the trace mapping \(\text{tr}_\Gamma : W^{1,q+1}(\Omega) \to W^{1-\frac{1}{q+1},q+1}(\Gamma)\)
- \(C_2^{tr}\) norm of the trace mapping \(\text{tr}_\Gamma : H^1(\Omega) \to H^{-1/2}(\Gamma)\)
- \(C_{X,Y}^{\Omega}\) norm of the embedding operator \(X(\Omega) \to Y(\Omega)\) between two function spaces over the domain \(\Omega\)
- \(C_P\) constant appearing in Poincaré’s inequality, see page 16
- \(C_1^\Omega = |\Omega|^{-\frac{q-1}{q+1}}, \text{ see page } 16\)
- \(C_2^\Omega = |\Omega|^{-\frac{q-1}{2(q+1)}}, \text{ see page } 16\)
- \(C(\varepsilon, r) = (r-1)\varepsilon^{-\frac{1}{1-r}}, \text{ constant appearing in Young’s inequality, see page } 17\)

**Notation related to shape optimization**

- \(u_d\) the desired acoustic pressure
- \(\mathcal{O}_{ad}\) set of admissible shapes, see page ??
- \(\chi_{\Omega_+}\) characteristic function of \(\Omega_+\)
- \(\text{Char}(\Omega)\) set of characteristic functions
- \(p\) the adjoint state
- \(\tau\) artificial time variable indicating varying subdomains
- \(h\) vector field over the domain \(\Omega\)
- \(F_\tau\) transformations of \(\overline{\Omega}\) into \(\mathbb{R}^d\)
- \(DF_\tau\) Jacobian of \(F_\tau\)
- \(A_\tau = (DF_\tau)^{-T}\)
- \(I_\tau = \det(DF_\tau)\)
- \(dJ(u, \Omega_+)h\) Eulerian derivative at \(\Omega_+\) in the direction of the vector field \(h\)
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