What is a random variable?

$(\Omega, \mathscr{F}, \mathbb{P})$ probability space

Real-valued **random variable** X is a $\mathscr{F}/\mathscr{B}(\mathbb{R})$-*measurable* mapping:

$$X\colon (\Omega, \mathscr{F}) \to (\mathbb{R}, \mathscr{B}(\mathbb{R})), \quad X^{-1}(B) \in \mathscr{F} \quad \forall B \in \mathscr{B}(\mathbb{R})$$

where $\mathscr{B}(\mathbb{R})$ Borel σ-algebra

X induces the **distribution** (pushforward measure) $\mathbb{P}_X$ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$

$$\mathbb{P}_X(B) := \mathbb{P}(\{\omega \in \Omega\colon X(\omega) \in B\}) = \mathbb{P}(X^{-1}(B)), \quad B \in \mathscr{B}(\mathbb{R})$$

Generalisation: Random elements

$(\Omega, \mathscr{F}, \mathbb{P})$ probability space; $(\Psi, \mathscr{G})$ measurable space

Ψ-valued **random element** X is a $\mathscr{F}/\mathscr{G}$-measurable mapping:

$$X\colon (\Omega, \mathscr{F}) \to (\Psi, \mathscr{G}), \quad X^{-1}(G) \in \mathscr{F} \quad \forall G \in \mathscr{G}$$

where $\mathscr{G}$ is a suitable σ-algebra

X induces the **distribution** $\mathbb{P}_X$ on $(\Psi, \mathscr{G})$:

$$\mathbb{P}_X(G) := \mathbb{P}(\{\omega \in \Omega\colon X(\omega) \in G\}) = \mathbb{P}(X^{-1}(G)), \quad G \in \mathscr{G}$$

Examples

- Random variable: $\Psi = \mathbb{R}$, $\mathscr{G} = \mathscr{B}(\mathbb{R})$ Borel σ-algebra

- Random vector: $\Psi = \mathbb{R}^n$, $\mathscr{G} = \mathscr{B}(\mathbb{R}^n)$

- $\Psi = W$ (separable) Banach space with norm $\|\cdot\|$, $\mathscr{G} = \mathscr{B}(W)$

  Distribution of X could be a Gaussian measure ...

- **Random field**: $\Psi = \{\text{functions } u\colon D \to \mathbb{R}\}$, $D \subset \mathbb{R}^d$
  Notation: $\Psi = \mathbb{R}^D$

  What is a suitable σ-algebra $\mathscr{G}$?

Alternative definition

A (real-valued) **random field** u is a family $\{u(x) \colon x \in D\}$ of real-valued random variables $u(x)$ with index space $D \subset \mathbb{R}^d$

Finite-dimensional distributions of u:
Familiy of distributions of all random vectors $[u(x_1), \ldots, u(x_n)]^\top$ where $x_1, \ldots, x_n \in D$

When is a family $\{u(x) \colon x \in D\}$ of real-valued random variables a $\mathbb{R}^D$-valued random element?

Families of random variables

Starting point:

$$u\colon \Omega \times D \to \mathbb{R}$$

where each $u(\cdot, x)$ is $\mathscr{F}/\mathscr{B}(\mathbb{R})$-measurable.

Goal: Construct σ-algebra $\mathscr{G}$ s.t. the mapping

$$\Omega \ni \omega \mapsto u(\omega, \cdot) \in \mathbb{R}^D$$

is $\mathscr{F}/\mathscr{G}$-measurable.

Families of random variables

Sufficient condition: $\mathscr{G}$ is the σ-algebra generated by the sets $\{f \in \mathbb{R}^D : [f(x_1), \ldots, f(x_n)]^\top \in B\}$ for any $B \in \mathscr{B}(\mathbb{R}^n)$, $x_1, \ldots, x_n \in D$
Notation: $\mathscr{C}(\mathbb{R}^D)$

Given a family of finite-dimensional distributions which satisfy certain consistency conditions there is a (unique) random field (interpreted as measurable mapping from $(\Omega, \mathscr{F})$ onto $(\mathbb{R}^D, \mathscr{C}(\mathbb{R}^D))$) with these finite-dimensional distributions.