Proof

$\underline{n=0}$: $u_0^{(j)} \in \mathcal{A}$ clear (by definition of $\mathcal{A}$)

$\underline{n \longrightarrow n+1}$ Recall $H = (0 \;\; I)$, $K_{n+1} = C_{n+1} H^T (H C_{n+1} H^T + \Gamma)^{-1}$

$$K_{n+1} = \begin{pmatrix} C_{n+1}^{uu} & C_{n+1}^{up} \\ (C_{n+1}^{up})^T & C_{n+1}^{pp} \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} \left((0 \;\; I) \begin{pmatrix} C_{n+1}^{uu} & C_{n+1}^{up} \\ (C_{n+1}^{up})^T & C_{n+1}^{pp} \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} + \Gamma \right)^{-1}$$

$$= \begin{pmatrix} C_{n+1}^{up} \\ C_{n+1}^{pp} \end{pmatrix} \left(C_{n+1}^{pp} + \Gamma \right)^{-1} = \begin{pmatrix} C_{n+1}^{up} \left(C_{n+1}^{pp} + \Gamma \right)^{-1} \\ C_{n+1}^{pp} \left(C_{n+1}^{pp} + \Gamma \right)^{-1} \end{pmatrix}$$

$$K_{n+1} H = \begin{pmatrix} 0 & C_{n+1}^{up} \left(C_{n+1}^{pp} + \Gamma \right)^{-1} \\ 0 & C_{n+1}^{pp} \left(C_{n+1}^{pp} + \Gamma \right)^{-1} \end{pmatrix}$$

$$z_{n+1}^{(j)} = \begin{pmatrix} I & -C_{n+1}^{up} \left(C_{n+1}^{pp} + \Gamma \right)^{-1} \\ X & \end{pmatrix} \begin{pmatrix} u_n^{(j)} \\ \mathcal{G}(u_n^{(j)}) \end{pmatrix}$$

$$+ \begin{pmatrix} C_{n+1}^{up} \left(C_{n+1}^{pp} + \Gamma \right)^{-1} y_{n+1}^{(j)} \\ X \end{pmatrix}$$

$$\Rightarrow u_{n+1}^{(j)} = u_n^{(j)} - C_{n+1}^{up} \left(C_{n+1}^{pp} + \Gamma \right)^{-1} \mathcal{G}(u_n^{(j)}) + C_{n+1}^{up} \left(C_{n+1}^{pp} + \Gamma \right)^{-1} y_{n+1}^{(j)}$$

$$= u_n^{(j)} + C_{n+1}^{up} \underbrace{\left(C_{n+1}^{pp} + \Gamma \right)^{-1} \left(y_{n+1}^{(j)} - \mathcal{G}(u_n^{(j)}) \right)}_{=: d_n^{(j)}}$$

Note that

$$C_{n+1}^{up} = \frac{1}{J} \sum_{k=1}^{J} \left((\hat{p}_n^{(k)})^T - \bar{p}_n^T \right) \underline{u_n^{(k)}}$$

$$\Rightarrow u_{n+1}^{(j)} = u_n^{(j)} + \frac{1}{J} \sum_{k=1}^{J} \left\langle \hat{p}_n^{(k)} - \bar{p}_n^T , d_n^{(j)} \right\rangle \underbrace{u_n^{(k)}}_{\in \mathcal{A}} \in \mathcal{A}$$

$$\Rightarrow u_{n+1} = \frac{1}{J} \sum_{j=1}^{J} \underbrace{u_{n+1}^{(j)}}_{\in \mathcal{A}} \in \mathcal{A} \qquad \square$$

Connection (cont.)

$$u_1 := \frac{1}{J}\sum_{j=1}^{J} u_1^{(j)} = \frac{1}{J}\sum_{j=1}^{J} u_0^{(j)} + C_1^{up}(C_1^{pp}+\Gamma)^{-1}\left(\frac{1}{J}\sum_{j=1}^{J}(y_1^{(j)} - Gu_0^{(j)})\right)$$

$$= m_J + C_1^{up}(C_1^{pp}+\Gamma)^{-1}\left(y + \frac{1}{J}\sum_{j=1}^{J}\eta_1^{(j)} - Gm_J\right)$$

$$\xrightarrow{J\to\infty} \bar{u} + CG^*(GCG^*+\Gamma)^{-1}(y - G\bar{u}) = u_{TP}$$