UQ Group

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Overview

1. Chapter 2
   - 2.2: Finite-dimensional inverse problems
   - 2.3: Underdetermined Problems
   - 2.4: Common Structure
2. Chapter 3

Classical inverse problem

Basic Problem

$$y = \mathcal{G}(u),$$

Solve for $u \in X$, given $y \in Y$, where X, Y are Banach spaces. $\mathcal{G}$ is called the *observation operator*. y is called the *data*.
This is often *ill-posed*!

Least-Squares Problem

$$\text{argmin}_{u \in X}\left(\frac{1}{2}\|y - \mathcal{G}(u)\|_Y^2\right)$$

Still often *ill-posed*!

Regularization

Regularized Problem

For some Banach space $(E, \|\cdot\|_E)$ contained in X, and point $m_0 \in X$,

$$\text{argmin}_{u \in E}\left(\frac{1}{2}\|y - \mathcal{G}(u)\|_Y^2 + \frac{1}{2}\|u - m_0\|_E^2\right)$$

If $(E, \|\cdot\|_E)$ and m_0 are chosen properly, could be well-posed; however, they are arbitrary without other assumptions.

It is advantageous to take a statistical approach.

Statistical Approach

$$y = \mathcal{G}(u) + \eta,$$

where η is a mean zero random variable called the *obeservational noise*.

Bayesian approach for finding u given y

1. Describe prior beliefs about u in terms of probability measure μ_0 (density π_0).
2. Use Bayes' formula to calculate posterior probabilty measure μ^y (density π^y), for u given y.

Statistical Approach

Assume $\eta \in \mathbf{R}^q$ is a random variable with density ρ, then

$$\rho(y|u) := \rho(y - \mathcal{G}(u)).$$

This is called the *data likelihood*.
By Bayes' formula

$$\pi^y(u) \propto \rho(y - \mathcal{G}(u))\pi_0(u).$$

By the Radon-Nikodym derivative

$$\frac{d\mu^y}{d\mu_0}(u) \propto \rho(y - \mathcal{G}(u)).$$

WLOG, can write as

$$\frac{d\mu^y}{d\mu_0}(u) \propto \exp(-\Phi(u; y)).$$

Not based on Lebesgue measure. Can generalize to infinite dimensions!

MAP estimator

Maximum a posteriori estimator: a point u which maximizes the posterior p.d.f. π^y.
If $\eta \sim \mathcal{N}(0, B)$ and $\mu_0 = \mathcal{N}(m_0, \Sigma_0)$, then the MAP estimator is

$$\text{argmin}_{u \in E}\left(\frac{1}{2}|y - \mathcal{G}(u)|_B^2 + \frac{1}{2}|u - m_0|_{\Sigma_0}^2\right).$$

1 Chapter 2

- 2.2: Finite-dimensional inverse problems
- 2.3: Underdetermined Problems
- 2.4: Common Structure

2 Chapter 3

2.3

- If $\mathcal{G}$ is linear, η is Gaussian, and μ_0 is Gaussian, μ^y is Gaussian.
- Overdetermined problems: μ^y goes to Dirac measure as η goes to zero.
- Underdetermined Problems: $u \in \mathbf{R}^n$, $y \in \mathbf{R}^q$, $n \geq q$: Prior measure plays important role in posterior, even as noise goes to zero.

Common Structure

1. u, y are in a separable Banach spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$.
2. μ_0 is Gaussian.
3. Relationship of posterior w.r.t. the prior:

$$\frac{d\mu^y}{d\mu_0}(u) = \frac{1}{Z(y)} \exp(-\Phi(u; y)).$$

4. Φ satisfies Assumption 2.6 (bounded from below, bounded from above, and Lipschitz in u and y).
5. (If $Y = \mathbf{R}^q$, Assumption 2.6 is satisfied if G satisfies Assumption 2.7).

Properties from Common Structure

- μ^y is well-defined
- μ^y depends continuously on y
- Desirable perturbation properties of μ^y based on finite-dimensional approximations of Φ and $\mathcal{G}$. (Important because we normally will be using a computer to find some finite-dimensional approximations to the forward problem.)

Examples

- Pointwise data for a random field
- Inverse problem for a diffusion coefficient
- Wave speed for the wave equation
- Initial condition for the heat equation
- Fluid mechanics (weather, oceanography)
- Subsurface geophysics
- Molecular dyanmics