Brownian Motion and the Dirichlet Problem

Mario Teixeira Parente

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Topics to be discussed

1. Solving the Dirichlet problem on bounded domains

2. Application: Recurrence/Transience of Brownian motion

3. Further topics
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What is the Dirichlet problem?

Assume $D$ is a connected open subset of $\mathbb{R}^n$. Find $h \in C^2(D) \cap C(\overline{D})$ with

\[
\begin{cases}
\Delta h = 0 & \text{in } D \\
h = f & \text{on } \partial D.
\end{cases}
\]

Remark: Continuity up to the boundary in this case means

\[
\lim_{\substack{x \to z \\
x \in D}} h(x) = f(z), \quad z \in \partial D.
\]
Our target

Consider Brownian motion \((B_t)_{t \geq 0}\) in \(n\) dimensions. Define

\[ \tau_D := \inf \{ t > 0 : B_t \in D^c \} \]

as the exit time of \(D\). \((\tau_D\) is a stopping time!\)

**Theorem**

Suppose \(D\) is bounded, every point on \(\partial D\) is regular and \(f : \partial D \rightarrow \mathbb{R}\) is continuous. Then the unique solution to the Dirichlet problem is given by

\[ h(x) := \mathbb{E}^x f(B_{\tau_D}). \]
What does Itô’s formula say?

Assume \( h \in C^2(\mathbb{R}^n) \) is a solution to the Dirichlet problem on \( D \) with boundary values \( f \).

Itô formula for Brownian motion:

\[
 h(B_t) = h(B_0) + \int_0^t \nabla h(B_s) \, dB_s + \frac{1}{2} \int_0^t \Delta h(B_s) \, ds
\]

... with stopping time \( \tau_D \):

\[
 h(B_{\tau_D \wedge t}) = h(B_0) + \int_0^{\tau_D \wedge t} \nabla h(B_s) \, dB_s + \frac{1}{2} \int_0^{\tau_D \wedge t} \Delta h(B_s) \, ds
\]

Martingale in \( t \)
What does Itô’s formula say? (II)

Let $x \in D$ and take expectations:

$$\mathbb{E}^x h(B_{\tau_D \wedge t}) = \mathbb{E}^x h(B_0) = h(x)$$

Since $\tau_D$ is finite and $h$ is bounded on $D$, letting $t \to \infty$ gives

$$h(x) = \mathbb{E}^x h(B_{\tau_D}) = \mathbb{E}^x f(B_{\tau_D}).$$
Steps for the theorem

We have to show:

1. \( h(x) := \mathbb{E}^x f(B_{\tau_D}) \) is harmonic in \( D \) (i.e. \( \Delta h = 0 \) in \( D \))

2. Continuity of \( h \) up to the boundary \( \partial D \):

\[
\lim_{x \to z, x \in D} h(x) = f(z), \quad z \in \partial D
\]

3. Uniqueness
Harmonicity of $h$

From PDE:

$h$ is harmonic in $D \iff h$ satisfies the mean value property in $D$

Mean value property:

$$h(x) = \int_{\partial B(x,r)} h(y) \sigma_{x,r}(dy) \text{ for every ball } B(x, r) \subset D$$

$\rightarrow$ Whiteboard
Harmonicity of $h$ (II)

We will need the *strong Markov property*. Our context is the *canonical model*, i.e.

$$
\Omega = C_0[0, \infty) := \{\omega : [0, \infty) \to \mathbb{R}^n : \omega \text{ is continuous and } \omega(0) = 0\}
$$

and $\mathbb{P}$ is chosen to be the probability measure such that

$$
B_t(\omega) := \omega(t)
$$

becomes a Brownian motion. Necessary ingredient: *Time-shift-operator* $\theta_s : \Omega \to \Omega, \omega \mapsto \omega(\cdot + s)$

This means

$$
\theta_s(\omega)(t) = \omega(t + s) = B_{t+s}(\omega).
$$
Harmonicity of $h$ (III)

We have a family of probability measures $(\mathbb{P}^x)_{x \in \mathbb{R}^n}$ with

$$\mathbb{P}^x (B_0 = x) = 1.$$ 

_Strong Markov property_ (our version): Let $X$ be a bounded random variable, $\tau$ be a stopping time and $x \in D$. Then

$$\mathbb{E}^x [X \circ \theta_\tau \mid \mathcal{F}_\tau] = \mathbb{E}^{B_\tau} X \quad \mathbb{P}^x \text{-a.s. on } \{\tau < \infty\}.$$ 

We define

$$\mathcal{F}_t := \bigcap_{s > t} \mathcal{F}_s^0,$$

where $\mathcal{F}_s^0$ denotes the smallest $\sigma$-algebra such that the mapping $\omega \mapsto \omega(s)$ (projection) is measurable for all $0 \leq s \leq t$. 

→ Whiteboard
Harmonicity of $h$ (IV)

Take a ball $B = B(x, r) \subset D$ and note that

$$B_{\tau_D} = B_{\tau_D} \circ \theta_{\tau_B}.$$

Proving the mean value property for $h(x) := \mathbb{E}^x f(B_{\tau_D})$:

$$h(x) = \mathbb{E}^x [\mathbb{E}^x [f(B_{\tau_D}) \circ \theta_{\tau_B} | \mathcal{F}_{\tau_B}]]$$

$$= f(B_{\tau_D})$$

SMP

$$= \mathbb{E}^x \left[ \mathbb{E}^{B_{\tau_B}} [f(B_{\tau_D})] \right]$$

$$= \mathbb{E}^x h(B_{\tau_B}) = \int_{\partial B(x,r)} h(y) \sigma_{x,r}(dy)$$
Continuity up to the boundary

Boundary has to be of some regularity:

\[ z \in \partial D \text{ regular} : \iff \mathbb{P}^z (\tau_D = 0) = 1 \]

Sufficient: Truncated cone condition

If there exists a truncated cone contained in \( D^c \) with vertex at \( z \), then \( z \) is regular.

Example:
Unique ness

Apply the Maximum principle for harmonic functions: Suppose $D$ is bounded and let $h \in C(\overline{D})$ be harmonic. Then

$$\max_{x \in \overline{D}} h(x) = \max_{x \in \partial D} h(x) \quad \text{and} \quad \min_{x \in \overline{D}} h(x) = \min_{x \in \partial D} h(x).$$

Take two solutions to the Dirichlet problem $h_1$ and $h_2$. Define $h := h_1 - h_2$ which is harmonic, $\in C(\overline{D})$ and $\equiv 0$ on $\partial D$. The Maximum principle gives

$$\max_{x \in \overline{D}} h(x) = \min_{x \in \overline{D}} h(x) = 0.$$

This implies $h \equiv 0$ on $\overline{D}$ and $h_1 = h_2$ on $\overline{D}$. 
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Example

Let \( n \geq 2 \), take \( 0 < r_1 < r_2 \) and consider

\[
D := \{ x \in \mathbb{R}^n : r_1 < |x| < r_2 \}
\]

with boundary values

\[
f(z) := \begin{cases} 
0 & \text{if } |z| = r_1, \\
1 & \text{if } |z| = r_2.
\end{cases}
\]
Example (II)

\[ h_1(x) := \mathbb{E}^x f(B_{\tau_D}) = \mathbb{P}^x(B_t \text{ exits } D \text{ through the outer boundary}) \]
solves the corresponding Dirichlet problem.
Example (III)

Typical examples of harmonic functions are also

\[ h_2(x) := \begin{cases} 
  a \log |x| + b & \text{if } n = 2, \\
  \frac{a}{|x|^{n-2}} + b & \text{if } n \geq 3.
\end{cases} \]

If we choose \( a, b \) correctly, \( h_2 \) also attains boundary values \( f \). By uniqueness,

\[ h_1 = h_2 \quad \text{on } D. \]
Example (IV)

We get

$$
\mathbb{P}^x \left( B_t \text{ exits } D \text{ through the outer boundary} \right) = \begin{cases} 
\frac{\log|x| - \log r_1}{\log r_2 - \log r_1} & \text{if } n = 2, \\
\left( \frac{r_2}{|x|} \right)^{n-2} \frac{|x|^{n-2} - r_1^{n-2}}{r_2^{n-2} - r_1^{n-2}} & \text{if } n \geq 3
\end{cases}
$$

$$
= \mathbb{P}^x \left( \tau_{r_2} < \tau_{r_1} \right),
$$

where $\tau_r$ denotes the hitting time of $\partial B(0, r)$. 
Example (V)

For (point) recurrence, let $r_1 \downarrow 0$ first:

$$\lim_{r_1 \downarrow 0} \mathbb{P}^x (\tau_{r_2} < \tau_{r_1}) = 1 \quad \text{for } 0 < |x| < r_2$$

Since $\lim_{r_1 \downarrow 0} \tau_{r_1} = \tau_0$, which is the hitting time of 0, it follows

$$\mathbb{P}^x (\tau_{r_2} < \tau_0) = 1 \quad \text{for } 0 < |x| < r_2.$$  

Finally, $\lim_{r_2 \uparrow \infty} \tau_{r_2} = \infty$ implies

$$\mathbb{P}^x (\tau_0 = \infty) = 1 \quad \text{for } x \neq 0.$$
Example (VI)

The Markov property gives the case $x = 0$: Note that

$$\{ \tau_0 < \infty \} = \{ \exists t > 0 : B_t = 0 \} = \bigcup_{n \in \mathbb{N}} \left\{ \exists t > \frac{1}{n} : B_t = 0 \right\}$$

and compute

$$\mathbb{P}^0 \left( \exists t > \frac{1}{n} : B_t = 0 \right) = \mathbb{P}^0 \left( \exists t > 0 : B_{t+\frac{1}{n}} = 0 \right)$$

$$= \mathbb{E}^0 \left[ \mathbb{P}^0 \left( \exists t > 0 : B_{t+\frac{1}{n}} = 0 \right) \bigg| \mathcal{F}_{1/n} \right]$$

$$\overset{\text{MP}}{=} \mathbb{E}^0 \left[ \mathbb{P}^{B_{1/n}} \left( \exists t > 0 : B_t = 0 \right) \right]$$

$$= \mathbb{E}^0 \left[ \mathbb{P}^{B_{1/n}} \left( \tau_0 < \infty \right) \right] = 0.$$
Example (VII)

Translation gives

\[ \mathbb{P}^x (\exists t > 0 : B_t = y) = 0 \quad \text{for all } x, y \in \mathbb{R}^n. \]

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For neighborhood recurrence/transience, let \( r_2 \uparrow \infty \) first and get for \( |x| > r_1 \)

\[ \mathbb{P}^x (\tau_{r_1} = \infty) = \lim_{r_2 \uparrow \infty} \mathbb{P}^x (\tau_{r_2} < \tau_{r_1}) = \begin{cases} 0 & \text{if } n = 2, \\ 1 - \left( \frac{r_1}{|x|} \right)^{n-2} & \text{if } n \geq 3. \end{cases} \]

\[ \implies \text{Brownian motion is neighborhood recurrent if } n = 2, \text{ but is neighborhood transient for } n \geq 3. \]
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Dirichlet problem on unbounded domains

Theorem

Suppose all points on $\partial D$ are regular and $f$ is a bounded and continuous function on $\partial D$. Then every bounded solution to the Dirichlet problem on $D$ with boundary values $f$ has the form

$$h(x) = \mathbb{E}^x [f(B_{\tau_D}), \tau_D < \infty] + c \mathbb{P}^x (\tau_D = \infty),$$

for some constant $c$. 
Poisson equation on bounded domains

Problem:

\[ \frac{1}{2} \Delta h = -f \quad \text{and} \quad \lim_{x \to z} h(x) = 0 \quad \text{for} \ z \in \partial D \]

Solution:

\[ h(x) = \mathbb{E}^x \int_0^{\tau_D} f(B_s) \, ds, \quad x \in D \]

Theorem

Let \( D \) be bounded and all points on \( \partial D \) be regular. Furthermore, let \( h \) be a \( C^2 \)-function on \( \mathbb{R}^n \) and \( f \) be a continuous function on \( \mathbb{R}^n \). Then

\[ (1) \iff (2). \]
References

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