Chapter 8

Structural Stability and Bifurcations

8.1 Smooth and Sudden Changes

So far, we discussed the behavior of solutions of a dynamical system

\[ \dot{x} = G(x), \quad x(0) = x^0, \quad (8.1) \]

under changes of the initial point \( x^0 \). Since practical problems modelled by (8.1) can be expected to involve uncertainties in the mapping \( G \), we should address also the question of the sensitivity of the solutions of (8.1) to perturbations of \( G \). Then, however, we may see a very different behavior, than in the case of changes of \( x^0 \).

By ODE-theory, the solution of (8.1) depends continuously on the initial point, as is obvious for the solution \( x(t) = \exp(tA)x^0 \) of the linear problem \( \dot{x} = Ax, \ x(0) = x^0 \). On the other hand, small changes of the mapping \( G \) may well produce abrupt changes in the solution behavior. We saw this already in the case of the discrete logistics problem

\[ x_{k+1} = g(x_k), \quad k = 0, 1, 2, \ldots, \quad g(x) := \mu x(1 - x), \quad \mu > 0. \quad (8.2) \]

Here, the fixed points are \( x^* = 0 \) and \( x^{**} = (\mu - 1)/\mu \), where \( x^* = 0 \) is asymptotically stable for \( 0 < \mu < 1 \) and unstable for \( \mu > 1 \), while \( x^{**} \) is unstable for \( 0 < \mu < 1 \) and asymptotically stable for \( 1 \leq \mu < 3 \). But, at \( \mu = 3 \) the solution behavior changes significantly: In fact, for any \( \mu > 3 \) the iterates tend to a 2-cycle consisting of the points

\[ z^\pm := \frac{1}{2\mu} [(\mu + 1) \pm \sqrt{(\mu + 1)(\mu - 3)}]; \]
that is, the attractor consisting of the single point \(x^{**}\) has become an attractor-set \(\{z^+, z^-\}\) of cardinality two.

Such sudden changes in the dynamics caused by smooth alterations of a system occur in many problems. Bridges bend continuously under an increasing load, but then suddenly break. As the saying goes, ‘there is one straw, that breaks the camels back’. Similarly, forces in a rock-structure can build up until friction no longer holds and the structure suddenly ruptures in an earthquake. A living cell suddenly changes its reproductive rhythm and doubles and redoubles cancerously. In each example, continuous changes suddenly lead to an abrupt change in the structure of the solution field. Accordingly, these situations are also called structural instabilities.

### 8.2 Structural Stability

In this section we outline – without proofs – the basic ideas behind the concept of structural stability. In essence, a dynamical systems (8.1) is structurally stable if its phase portrait remains topologically invariant under small perturbations of the mapping \(G\). This is illustrated by the two pairs of pictures in Figures 8.1 and 8.2. The two phase portraits 8.1 can be continuously deformed into each other, while, in the second pair, the connection between the two saddles is severed and thus there can no longer be a such a deformation.

![Figure 8.1: Conjugate](image1)

![Figure 8.2: Not conjugate](image2)

A precise definition of structural stability builds on the concept of topological equivalence of mappings. If \(x \mapsto Xx\) is a coordinate transformation on \(\mathbb{R}^n\) defined by some nonsingular \(X \in \mathbb{R}^{n \times n}\), then any matrix \(A \in \mathbb{R}^{n \times n}\) transforms into \(X^{-1}AX\). Analogously, suppose that \(\mathbb{R}^n\) is mapped onto itself by a homeomorphism \(h : \mathbb{R}^n \rightarrow \mathbb{R}^n\); i.e., a bijective, continuous mapping with a continuous inverse, then a nonlinear mapping \(f : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is transformed into the mapping \(g = h^{-1}fh\). Accordingly, the mappings \(f\) and \(g\) are
called topologically equivalent, if for some homeomorphism \( h \) on \( \mathbb{R}^n \) the following diagram is commutative:

\[
\begin{array}{ccc}
\mathbb{R}^n & f & \mathbb{R}^n \\
\downarrow h & & \downarrow h \\
\mathbb{R}^n & g & \mathbb{R}^n
\end{array}
\]

For the discussion of the topological equivalence of phase portraits of dynamical systems it is customary to use vector fields. Recall, that for a \( C^1 \)-mapping \( G : \mathbb{R}^n \rightarrow \mathbb{R}^n \), the unique solution of (8.1) passing through \( x \in \mathbb{R}^n \) has at \( x \) the tangent vector \( G(x) \). Accordingly, we will view \( G \) as defining a \( C^1 \) vector field on \( \mathbb{R}^n \) to be denoted by \( (\mathbb{R}^n, G) \). The solutions of (8.1) are the orbits of this vector field.

Two vector fields \( (\mathbb{R}^n, G_1), (\mathbb{R}^n, G_2) \) are topologically equivalent near \( x^1, x^2 \in \mathbb{R}^n \), if there exist neighborhoods \( U_1, U_2 \subset \mathbb{R}^n \) of \( x^1, x^2 \) and a homeomorphism \( h : U_1 \rightarrow U_2 \) with \( h(x^1) = x^2 \), which maps each orbit of the first vector field in \( U_1 \) onto an orbit of the second vector field in \( U_2 \), preserving the direction of time.

A linear vector field \( (\mathbb{R}^n, A) \) is hyperbolic, if \( A \in \mathbb{R}^{n \times n} \) has no eigenvalues with zero real part. Here, the next result shows, that a characteristic quantity is the number of eigenvalues with negative real part, called the index of \( A \):

8.2.1. The hyperbolic (linear) vector fields \( (\mathbb{R}^n, A) \) and \( (\mathbb{R}^n, B) \) are topologically equivalent if and only if \( A \) and \( B \) have the same index.

A proof will not be given; it involves consideration of certain bases of generalized eigenvectors corresponding to the two matrices. As a simple example, let

\[
A = \begin{pmatrix} -1 & -3 \\ -3 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}.
\]

Then we have

\[
B = XAX^{-1}, \quad X = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
\]

and hence \( X \exp(tA) = \exp(tB)X \). Since \( x \mapsto Xx \) is a homeomorphism on \( \mathbb{R}^2 \), the vector fields \( (\mathbb{R}^2, A) \) and \( (\mathbb{R}^2, B) \) are indeed topologically equivalent.

As indicated, structural stability of a vector field \( (\mathbb{R}^n, G) \) requires, that all sufficiently small perturbation of \( G \) produce topologically equivalent vector fields. For this we have to specify what perturbations are allowed. In the following definition we work with the normed linear space \( (C^1(\Omega), \| \cdot \|_1) \) of all continuously differentiable mappings \( G : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) on a given open set \( \Omega \), with the norm

\[
\|G\|_1 := \sup_{x \in \Omega} \|G(x)\|_2 + \sup_{x \in \Omega} \|DG(x)\|_2, \quad \forall G \in C^1(\Omega).
\]
8.2.2 Definition. For $G^0 \in C^1(\Omega)$ the vector field $(\Omega, G_0)$ is structurally stable in $(C^1(\Omega), \| \cdot \|_1)$, if there is an $\epsilon > 0$, such that for all $G \in C^1(\Omega)$ with $\| G - G^0 \|_1 < \epsilon$, the vector fields $(\Omega, G^0)$ and $(\Omega, G)$ are topologically equivalent.

We will not enter into a further study of this concept. The theory quickly reaches a considerable mathematical depth.

For a linear vector field $(\mathbb{R}^n, A)$ defined by some matrix $A$ it is natural to restrict the perturbations to the normed linear space $(\mathbb{R}^{n \times n}, \| \cdot \|_2)$. Recall that the eigenvalues of $A \in \mathbb{R}^{n \times n}$ depend continuously on the elements of the matrix. Hence, if the vector field $(\mathbb{R}^n, A)$ is hyperbolic, then there exists $\epsilon > 0$, such that for each $B \in \mathbb{R}^{n \times n}$ with $\| B - A \|_2 < \epsilon$, the vector field $(\mathbb{R}^n, B)$ is also hyperbolic. Moreover, it can be shown, that, for sufficiently small $\epsilon$, the index of any such $B$ is the same as that of $A$. Hence 8.2.1 implies, that the vector fields of $A$ and $B$ are topologically equivalent. This leads to the following result:

8.2.3. A linear vector field $(\mathbb{R}^n, A)$ is structurally stable in $(\mathbb{R}^{n \times n}, \| \cdot \|_2)$ if and only if $A$ is hyperbolic.

For nonlinear mappings the Hartman-Grobman theorem provides a connection with this linear result:

8.2.4. Let $\Omega \subset \mathbb{R}^n$ be an open set and for given $G \in C^1(\Omega)$ denote by $\Phi$ the flow of the dynamical system (8.1). Suppose, that $x^* \in \Omega$ is a hyperbolic equilibrium of (8.1). Then there exists a homeomorphism $h$ of an open neighborhood $U_1 \subset \Omega$ of $x^*$ onto an open neighborhood $U_2 \subset \mathbb{R}^n$ of the origin, such that $h(x^*) = 0$ and

$$h(\Phi(t, x)) = \exp(tDG(x^*)) h(x) \quad \forall t \in \mathcal{J}, \ x \in U_1,$$

where the open interval $\mathcal{J}$, $0 \in \mathcal{J}$, is locally determined.

Under the conditions of the theorem, (8.3) implies that the homeomorphism $h$ maps each solution of (8.1) in the neighborhood $U_1$ of $x^*$ onto a solution of the linearized system

$$\dot{y} = DG(x^*)y, \quad y(0) = h(x),$$

in the neighborhood $U_2$ of the origin and preserves the $t$-orientation. In other words, locally near the hyperbolic equilibrium $x^*$, the vector field $(\mathbb{R}^n, G)$ is topologically equivalent with the hyperbolic linear vector field $(\mathbb{R}^n, DG(x^*))$ near the origin.
8.3 Introduction to Bifurcations

Bifurcation theory is the mathematical study of changes in the topological properties of the phase portraits of a family of dynamical systems. We sketch only some elementary ideas underlying this large topic.

Many physical, chemical, and biological problems lead to parameter-dependent dynamical systems

\[ \dot{x} = G(x, \mu), \quad x(0) = x^0, \quad G : \Omega \subset \mathbb{R}^n := \mathbb{R}^m \times \mathbb{R}^d \longrightarrow \mathbb{R}^n, \quad n = m + d, \quad (8.4) \]

where \( G \in C^1(\Omega) \) on an open set \( \Omega \). Then, in essence, a bifurcation occurs at a parameter value \( \mu_0 \), if there are values of \( \mu \) arbitrarily close to \( \mu_0 \), for which the phase portraits of (8.4) are no longer topologically equivalent with the one for \( \mu_0 \). In other words, bifurcations identify the failure of the structural stability of the vector field \((\Omega, G(\cdot, \mu))\) for specific parameter values.

For instance, the planar system

\[ \dot{x} = A(\mu)x, \quad A(\mu) := \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}, \quad (8.5) \]

has the origin as an equilibrium for each value of \( \mu \). For \( \mu \neq 0 \) this equilibrium is hyperbolic and hence, by 8.2.3, the linear vector field \((\mathbb{R}^2, A(\mu))\) is structurally stable. More specifically, for \( \mu < 0 \) the phase portraits are asymptotically stable spirals and for \( \mu > 0 \) they are unstable spirals (see Figure 8.3). But for \( \mu = 0 \) the origin is a center point and no homeomorphism can map the periodic orbits for \( \mu = 0 \) onto the spiral trajectories for any nonzero \( \mu \), no matter how small. In other words, we have a bifurcation at \( \mu = 0 \).

![Figure 8.3: Structural instability](image)

Bifurcation theory develops strategies for investigating the types of bifurcations occurring within a family of dynamical systems, and for classifying and naming the different patterns. One aim is to produce subdivisions of the parameter space into regions induced by the topological equivalence of the corresponding phase portraits. Then bifurcations occur at points, that do not lie in the interior of one of these regions. A bifurcation diagram is a map of these regions, together with representative phase portraits for each of them.
Suppose that the system (8.4) involves a scalar parameter, i.e., that \( d = 1 \). If, say, \( G(x^*, \mu^*) = 0 \), and \( x^* \) is a hyperbolic equilibrium, then the vector field \( (\mathbb{R}^n, G(\cdot, \mu^*)) \) is structurally stable near \( x^* \). Generically, there are only two ways in which variation of the parameter can lead to a violation of the hyperbolicity condition, namely

- for some value of the parameter, a simple real eigenvalue \( \lambda \) of \( D_x G(x, \mu) \) becomes zero, or
- a pair of simple complex eigenvalues passes the imaginary axis and on the axis we have \( \lambda_{1,2} = \pm i\gamma, \gamma > 0 \).

It can be shown, that a higher dimensional parameter space is needed to allocate additional eigenvalues on the imaginary axis.

The two cases correspond to some basic types of bifurcations. A typical approach for characterizing such types is the use of certain normal forms for model systems, that exemplify the topological pattern of the phase portraits near the bifurcation.

**Limit Points:** These are also called turning points or saddle-node bifurcations. They are characterized by the collision and annihilation of two equilibria. A normal form is

\[
\dot{x} = x^2 + \mu, \quad (x, \mu) \top \in \mathbb{R}^2.
\] (8.6)

Hence equilibria exist only for \( \mu \leq 0 \) and they lie on a smooth curve in \( \mathbb{R}^2 \), namely, the parabola of Figure 8.4.

For \( \mu < 0 \) the equilibria are hyperbolic and the upper branch \( x^+(\mu) := \sqrt{-\mu}, \mu < 0 \), of the parabola is unstable, while the lower branch \( x^- (\mu) := -\sqrt{-\mu}, \mu < 0 \), is asymptotically stable. When \( \mu \) crosses zero from negative to positive values, the two equilibria collide and then disappear for \( \mu > 0 \). The term ”collision” is here appropriate, since the velocity \( (d/d\mu) x^\pm (\mu) \) of the approach tends to infinity as \( \mu \to 0 \). The limit point \( (0, 0) \top \in \mathbb{R}^2 \) is
characterized by the property, that the tangent of the equilibrium curve at this point is orthogonal to the parameter axis.

The bifurcation can also occur in reverse. The system
\[ \dot{x} = x^2 - \mu, \quad (x, \mu)^\top \in \mathbb{R}^2, \]
has no equilibria for \( \mu < 0 \) and two of them for \( \mu > 0 \).

The indicated name ’saddle-node bifurcation’ becomes clearer, if we consider (8.6) in the planar form
\[ \dot{x} = \begin{pmatrix} x_1^2 + \mu \\ -x_2 \end{pmatrix}, \quad (x, \mu)^\top \in \mathbb{R}^3. \tag{8.7} \]
For \( \mu < 0 \) the two equilibria are now a stable node and a saddle, and they are annihilated at \( \mu = 0 \), as shown in Figure 8.5 (where \( \beta \) stand for \( \mu \)).

**Figure 8.5: Saddle-node bifurcation**

**Hopf Bifurcations:** The characteristic feature of these bifurcations is the emergence of a periodic orbit from a stable equilibrium or vice versa. A normal form is
\[ \dot{z} = (\mu + i)z - |z|^2 z, \quad z = x_1 + ix_2 = \rho e^{i\theta}, \]
or in the decoupled form
\[ \begin{align*}
\dot{\rho} &= \rho(\mu - \rho^2) \\
\dot{\theta} &= 1.
\end{align*} \tag{8.9} \]
For \( \mu < 0 \) the phase portrait consists of asymptotically stable spirals, as shown in Figure 8.6 with \( \mu = -1 \). For \( \mu = 0 \) both eigenvalues pass through the imaginary axis with the velocity \( d\lambda_{1,2}/dt = 1 \). For \( \mu > 0 \) the first equation of (8.9) has the unstable equilibrium...
\( \rho_1 = 0 \) and the asymptotically stable equilibrium \( \rho_2(\mu) = \sqrt{\mu} \). For the full system (8.9) the latter defines a limit cycle with radius \( \sqrt{\mu} \), as shown in Figure 8.7 with \( \mu = 1 \):

Note that, generally, a limit cycle is an isolated closed orbit for which neighboring trajectories are not closed. In other words, the orbits of any center point are not limit cycles.

As before, the bifurcation can also be reversed. The system

\[
\dot{x} = A(\mu)x + (x^T x)x, \quad (x, \mu)^T \in \mathbb{R}^3, \quad (8.10)
\]

has for \( \mu < 0 \) an unstable limit cycle, which disappears when \( \mu \) crosses zero from negative to positive values and then turns into unstable spirals when \( \mu > 0 \). The bifurcations in (8.8) and (8.10) are sometimes called supercritical and subcritical Hopf bifurcations, respectively. This terminology is somewhat misleading, since "super" means here "after" the bifurcation point, and "sub" "before" the point, which, however, depends on the chosen direction of the parameter variation.

We followed here the wide-spread practice of using the term "Hopf-bifurcation" for the emergence of a periodic orbit from a stable equilibrium as a parameter crosses a critical value. But the subject has its origins in the work of Poincare around 1892, followed by extensive studies by Andronov and coworkers around 1930, while the fundamental paper of E. Hopf appeared in 1942. Accordingly, the term Poincare-Andronov-Hopf bifurcation is thought to be more justified by many.

**Transcritical Bifurcation:** A normal form is

\[
\dot{x} = x(\mu - x), \quad (x, \mu)^T \in \mathbb{R}^2. \quad (8.11)
\]

Thus the equilibria form two intersecting straight lines in \( \mathbb{R}^2 \), as shown in Figure 8.8 (where \( \lambda \) stands for \( \mu \)). At the crossing point \((0,0)^T\) these equilibria exchange their
\[ \frac{dx}{dt} = (\lambda - x)x \]

Figure 8.8: Transcritical bifurcation

stability properties; that is, the unstable equilibria become asymptotically stable and vice versa.

Note, that beyond the bifurcation point the number of equilibria has not changed, while in the case of limit points two equilibria appear or disappear.

**Cusp Bifurcation:** This is also called a pitchfork bifurcation or the occurrence of a branch point. A normal form corresponds with the first equation of (8.9); i.e.,

\[ \dot{x} = x(\mu - x^2), \quad (x,\mu)^\top \in \mathbb{R}^2. \]  

(8.12)

Evidently, \( x^* = 0 \) is an equilibrium for all \( \mu \), which is asymptotically stable for \( \mu < 0 \) and unstable for \( \mu > 0 \). Then, for \( \mu > 0 \) two additional, asymptotically stable equilibria \( x^\pm(\mu) = \pm \sqrt{\mu} \) appear, as shown in Figure 8.9.

Thus, in this case, when \( \mu \) passes through zero from negative to positive values, the asymptotically stable equilibrium \( x^* = 0 \) becomes unstable and two curves of asymptotically stable equilibria are branching off.

As before, this bifurcation can also occur in reverse. The system

\[ \dot{x} = x(\mu + x^2), \quad (x,\mu)^\top \in \mathbb{R}^2, \]

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has for $\mu < 0$ the stable equilibrium $x^* = 0$ and the two unstable equilibria $x^\pm = \pm \sqrt{-\mu}$. Then, for $\mu > 0$ only the unstable equilibrium $x^* = 0$ remains.

## 8.4 Two Examples

### 8.4.1 A Simple Mechanical Example

As an illustration of the appearance of limit points and branch points, we consider the simple framework of Figure 8.10, where the two straight, rigid, and pin-jointed rods are subjected to two (dead)-loads $\mu$ and $\nu$. The deformation is specified by the angle $y$ and, without loads, the (linear) spring forces the rods into the horizontal reference configuration, where $y = 0$.

![Figure 8.10: A simple framework](image.png)

The angle $y$ corresponds to a lengthening of the spring by $2y$, which therefore has the elastic energy $(1/2)k(2y)^2$. Here $k$ denotes the spring constant, which we set to one. Up to constants, the potential energy due to the forces $\mu$ and $\nu$ equals $\mu(2\cos y)$ and $\nu \sin y$, respectively. Thus at a given state $x := (y, \mu, \nu)^\top$ in the open set

$$\Omega := \{(y, \mu, \nu)^\top : |y| < \frac{\pi}{2}, \mu, \nu \in \mathbb{R}^1\},$$

the system has the total potential energy

$$V = V(y, \mu, \nu) := \frac{1}{2}(2y)^2 + 2\mu \cos y - \nu \sin y, \quad (y, \mu, \nu)^\top \in \Omega,$$

and the equilibria are the solutions of the parameterized nonlinear equation

$$G(y, \mu, \nu) := D_yV(y, \mu, \nu) = 2y - 2\mu \sin y - \nu \cos y, \quad (y, \mu, \nu)^\top \in \Omega.$$

The solution set

$$M := \{(y, \mu, \nu)^\top \in \Omega : G(y, \mu, \nu) = 0\}$$

is the 2-dimensional surface in $\mathbb{R}^3$ shown in Figure 8.11. For further insight, Figure 8.12 depicts some contour lines corresponding to constant loads $\nu$ in the plane of the variables $y, \mu$. Evidently, $M$ has a saddle-point at $(0, 1, 0)^\top$. 

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Theoretical mechanics provides, that a point on the surface is stable or unstable if the second derivative of the potential energy $V$; i.e.,

$$D_{yy}V(y, \mu, \nu) := 2 - 2\mu \cos y + \nu \sin y,$$  \hspace{1cm} (8.17)

is positive or negative, respectively. Hence, the solutions of the system of equations

$$\begin{pmatrix} G(y, \mu, \nu) \\ D_yG(y, \mu, \nu) \end{pmatrix} = 0, \quad (y, \mu, \nu) \in \Omega,$$  \hspace{1cm} (8.18)

define the stability boundary of the system.

We fix the load $\nu = \nu_0$ and allow only $\mu$ to vary. In other words, we consider the reduced equation

$$G_{\nu_0}(y, \mu) := G(y, \mu, \nu_0) = 0.$$  \hspace{1cm} (8.19)

As Figure 8.12 indicates, for $\nu_0 \neq 0$ the solution set of (8.19) consists of two smooth curves on $M$ corresponding to positive and negative values of $y$, respectively. Let $t \rightarrow (y(t), \mu(t))$ be a (smooth) parametric representation of either one of these curves, then, by differentiation, it follows that

$$D_yG(y(t), \mu(t), \nu_0) \ y'(t) + D_\mu G(y(t), \mu(t), \nu_0) \ \mu'(t) = 0.$$  \hspace{1cm}

Hence, by (8.18), a point of the curve is on the stability boundary if

$$D_\mu G(y(t), \mu(t), \nu_0) \ \mu'(t) = -2 \sin y(t) \ \mu'(t) = 0, \quad (y(t), \mu(t), \nu_0)^\top \in \Omega.$$  \hspace{1cm} (8.20)

This implies that $\mu'(t) = 0$ and, therefore, that, at a point on the intersection of the curve with the stability boundary, the tangent must be perpendicular to the $\mu$-axis. Accordingly, these points are limit points with respect to $\mu$. Figure 8.12 indicates, that that there exists
exactly one limit point for each \( \nu_0 \neq 0 \). The line of the limit points defines the stability boundary sketched in Figure 8.12.

Suppose that we start at a point on the curve for \( \nu_0 = -0.1 \) with \( y = 1 \) and some \( \mu > 1 \). Then a gradual decrease of the load \( \mu \) reduces the state-angle \( y \) until we reach the limit point of the particular curve and hence the stability boundary. Beyond this point the equilibria on the curve are unstable, and physical considerations suggest, that the frame will snap through to a point with \( y < 0 \) on the corresponding stable branch for the load \( \nu_0 = -0.1 \). This is sketched by the dotted line in Figure 8.12. Of course, this jump cannot be derived from our static equilibrium model. After the snap-through, a continuing decrease of the load \( \mu \) causes the system to follow the stable load path.

In the case \( \nu_0 = 0 \), the solutions of the reduced equation (8.19) form the four branches

\[
\begin{align*}
(a) \quad y &= 0, \quad \mu < 1, \\
(b) \quad y &= 0, \quad \mu > 1, \\
(c) \quad 0 < y < \frac{\pi}{2}, \quad \mu = y / \sin(y), \\
(d) \quad 0 > y > -\frac{\pi}{2}, \quad \mu = y / \sin(y),
\end{align*}
\]

which meet at the saddle point \( y^* = (0, 1, 0)^T \) of the surface. Evidently, this is the only point where the stability boundary intersects the four curves, and, as Figure 8.12 shows, at this point we have a pitchfork bifurcation. If we are, say, at the origin of Figure 8.12, then the rods are in a straight line, and there are no applied loads. With increasing \( \mu \) the system follows branch (a); i.e., the state \( y \) remains zero until we reach \( \mu = 1 \). The continuing branch (b) is unstable and hence an increase of \( \mu \) beyond 1 will cause the system to depart from the straight position and to follow either branch (c) or (d). In practice, the specific choice is determined by small, inevitable perturbing forces.

### 8.4.2 Hopf Bifurcation in the Lorenz System

The well known ODE-system developed by E. N. Lorenz is a simplified model of certain convection phenomena in the earth’s atmosphere and has the form

\[
\dot{x} = G(x, \mu), \quad G(x, \mu) := \begin{pmatrix}
a(x_2 - x_1) \\
x_1(\mu - x_3) - x_2 \\
x_1x_2 - bx_3
\end{pmatrix}, \quad (x, \mu)^T \in \mathbb{R}^4.
\]

Here, the parameter \( \mu \) denotes a (relative) Raleigh number and the constants are often set to \( a = 10 \) and \( b = 8/3 \). Evidently, the origin \( x^* = 0 \) of \( \mathbb{R}^3 \) is an equilibrium for all \( \mu \) and, it is easily checked, that for \( \mu \geq 1 \) there are the two additional equilibria

\[
x^{\pm}(\mu) := (\pm \gamma(\mu), \pm \gamma(\mu), \mu - 1)^T, \quad \gamma(\mu) = \sqrt{b(\mu - 1)}.
\]

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The spectrum of
\[ D_xG(x, \mu) = \begin{pmatrix} -a & a & 0 \\ \mu - x_3 & -1 & -x_1 \\ 0 & x_1 & -b \end{pmatrix}, \]  
(8.23)
at \( x = x^* \) equals
\[ \sigma(D_xG(x^*, \mu)) = \{-b\} \cup \sigma\left(\begin{pmatrix} -a & a \\ \mu & -1 \end{pmatrix}\right) \]
\[ = \{-b, \frac{1}{2}[-(a + 1) \pm \sqrt{(a + 1)^2 + 4a(\mu - 1)}]\}. \]

Thus \( x^* \) is asymptotically stable for \( \mu < 1 \). At \( \mu = 1 \) one eigenvalue becomes zero and \((0, 0, 1)^T\) is a cusp point, where the origin becomes unstable and for \( \mu > 1 \) the two equilibrium curves (8.22) branch off. Their projections from \( \mathbb{R}^3 \) onto the \((x_1, \mu)\)-plane are shown in Figure 8.13:

![Figure 8.13: Equilibria of the Lorenz equations](image)

A short calculation shows, that the negative characteristic polynomial of \( D_xG(x^\pm(\mu), \mu) \) equals
\[ p_\mu(\lambda) := c_0 + c_1 \lambda + c_2 \lambda^2 + \lambda^3, \quad c_0 = 2ab(\mu - 1), \quad c_1 = b(a + \mu), \quad c_2 = a + b + 1 \]  
(8.24)

A method for computing Hopf bifurcations by J. Guckenheimer, M. Myers, and B. Sturmfels (SIAM J. Num. Anal 34, 1997, 1-21) is based on the observation, that any (monic) polynomial \( q \) has a nonzero pair of roots \( \lambda \) and \(-\lambda\) if and only if \( \lambda \) is a common root of the polynomials \( q(\lambda) + q(-\lambda) \) and \( q(\lambda) - q(-\lambda) \). In our case, this implies that (8.24) has a nonzero root-pair exactly if there exists a solution \( z = \lambda^2 \) of the two equations
\[ c_0 + c_2 z = 0, \quad c_1 + z = 0 \]
This requires that \( 0 = c_0 - c_1 c_2 = 2ab(\mu - 1) - b(a + b + 1)(a + \mu) \), and hence that
\[ \mu = \mu_h := \frac{a(3 + a + b)}{a - b - 1}. \]
Indeed, the eigenvalues of $D_x G(x^\pm(\mu_h), \mu_h)$ are
\[
\lambda_1 = -(a + b + 1), \quad \lambda_{2,3} = \pm i \sqrt{\frac{2ab(a + 1)}{a - b - 1}}.
\]
A closer study of the characteristic polynomial (8.24) shows, that the equilibria $x^\pm(\mu)$ are asymptotically stable for $1 < \mu < \mu_h$ and unstable for $\mu > \mu_h$. For $\mu = \mu_h$ we expect Hopf bifurcations at the points $(x^\pm(\mu_h), \mu_h)^\top$ on the two branches. The details of the situation are more complicated and are discussed in a monograph by C. Sparrow, (Springer, 1982).

8.5 Some Further Aspects of Bifurcation Theory

The bifurcations discussed in Section 8.3 involved systems with a scalar parameter and, in each case, the bifurcation type was solely determined by properties of the system at the particular equilibria. This need not always be the so; in fact, one has to distinguish between two principal classes of bifurcations:

- **Local bifurcations**, which can be analysed entirely in terms of the local stability properties of equilibria as parameters cross through critical thresholds.
- **Global bifurcations**, which involve topological changes in the behavior of the trajectories in phase space extending over larger distances and involving, e.g., interrelated changes at several equilibria.

For an example of a global bifurcation consider the system
\[
\dot{x} = G(x, \mu) := \begin{pmatrix} x_2 \\ x_1 + \mu x_2 - x_2^2 \end{pmatrix}, \quad (x, \mu)^\top \in \mathbb{R}^2 \times \mathbb{R}^1. \tag{8.25}
\]
For each $\mu$, the system has the same two equilibria, $x^* := (0, 0)^\top$ and $x^{**} := (1, 0)^\top$.

Because of
\[
D_x G(x, \mu) = \begin{pmatrix} 0 & 1 \\ 1 - 2x_1 & \mu \end{pmatrix},
\]
$x^*$ is always a saddle, while $x^{**}$ changes from a spiral sink for $\mu < 0$ to a spiral source for $\mu > 0$. The phase portraits for the cases $\mu = 0$, $\mu = -1$ and $\mu = 1$ are shown in Figures 8.14, 8.15, and 8.16.

For $\mu = 0$ there exists a trajectory, that both emanates and terminates at $x^*$. This is a so called homoclinic orbit and $x^*$ is said to be a homoclinic point. The region bounded by the homoclinic orbit forms a neighborhood of the equilibrium $x^{**}$ and contains periodic orbits around that point. Thus, locally, $x^{**}$ is a stable center point. Evidently, the eigenvalues
of $D_x G(x^{**}, 0)$ are $\pm i$ and hence we might expect a Hopf bifurcation for $\mu = 0$. But the homoclinic orbit is formed at the other equilibrium $x^*$ and hence is not a local feature of $x^{**}$. In other words, the solution behavior is only partially determined by the eigenvalues at that equilibrium and depends on the entire ODE.

So far we considered only problems with a scalar parameter. For higher dimensional parameter spaces the phenomena quickly become more complicated and even an introductory discussion would exceed our framework. We expand only on one aspect, which arose in connection with the mechanical example in subsection 8.4.1. This example involved a two-dimensional parameter vector and, as Figure 8.11 showed, the equilibria formed a two-dimensional surface in $\mathbb{R}^3$.

In essence, the existence of such equilibrium manifolds depends on the number of parameters in the equations.

As before, consider a parameter-dependent dynamical system

$$
\dot{x} = G(x, \mu), \quad x(0) = x^0, \quad G : \Omega \subset \mathbb{R}^n := \mathbb{R}^m \times \mathbb{R}^d, \quad n = m + d, \quad (8.26)
$$
defined by a $C^1$ mapping $G$ on an open set $\Omega$. Then our interest centers on conditions under which the set of all equilibria

$$M := \{(x, \mu) \in \mathbb{R}^n : G(x, \mu) = 0\}$$

(8.27)
is a differentiable manifold.

We will not enter into the theory of manifolds, but use – in place of a definition – the following sufficient characterization of submanifolds of $\mathbb{R}^n$:

**8.5.1.** Let $F : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, $n > m$, be continuously differentiable on the open set $\Omega$ and assume that the zero set $M = \{x \in \Omega : F(x) = 0\}$ is not empty. If $\text{rank} \, DF(x) = m$ for all $x \in M$, then $M$ is a $C^1$-submanifold of $\mathbb{R}^n$ of dimension $d = n - m$.

Generally, the mapping $F$ is a submersion at $x \in \Omega$ if $DF(x) \in \mathbb{R}^m \times \mathbb{R}^n$ has maximal rank $m$ and hence maps onto $\mathbb{R}^m$. Thus, $M$ is a submanifold of $\mathbb{R}^n$, if $F$ is a submersion at each point of this set. Accordingly, **8.5.1** is also called the submersion theorem.

A simple example is the Euclidean unit sphere

$$S^{n-1} := \left\{ x \in \mathbb{R}^n : x^T x = 1 \right\},$$

(8.28)
centered at the origin. It is the zero set of $x \in \mathbb{R}^n \mapsto F(x) := x^T x - 1$, and, since $DF(x) = \text{rank}(2x_1, \ldots, 2x_n) = 1$ for $x \in S^{n-1}$, it follows from **8.5.1** that $S^{n-1}$ is an $(n-1)$-submanifold of $\mathbb{R}^n$.

For the normal form (8.6) of a limit-point the derivative $DG(x, \mu) = (2x, 1)$ has rank one for all $(x, \mu)^T \in \mathbb{R}^2$. Thus, by **8.5.1**, the equilibrium set $M$ is a one-dimensional submanifold of $\mathbb{R}^2$, namely the parabola of Figure 8.4.

The situation is different for the set

$$M = \{(0, \mu)^T, \forall \mu \in \mathbb{R}^1\} \cup \{(-\sqrt{\mu}, \mu)^T, \forall \mu \geq 0\} \subset \mathbb{R}^2,$$

(8.29)
of all equilibria of the cusp bifurcation (8.12) shown in Figure 8.9. Here, we have $DG(x, \mu) = (\mu - 3x^2, x)$ and hence rank $DG(0, 0) = 0$; i.e., $G$ is not a submersion at the bifurcation point $(0, 0)^T \in M$. A $d$-dimensional manifold is a topological space, where each point necessarily has an open neighborhood, which is homeomorphic with an open subset of $\mathbb{R}^d$. But, clearly, in (8.29) the bifurcation point $(0, 0)^T \in M$ has no neighborhood in $M$, which is homeomorphic with an open interval of $\mathbb{R}^1$. In other words, the entire set (8.29) is not a one-dimensional submanifold of $\mathbb{R}^2$. In a sense, the problem is here, that the equation (8.12) does not include enough parameters. If, instead, the two-parameter system

$$\dot{x} = G(x, \mu, \nu) := \nu + \mu x - x^3, \quad x \in \mathbb{R}^1 \quad \mu, \nu \in \mathbb{R}^1$$

(8.30)
is used, then rank \(DG(x, \mu, \nu) = 1\) on all of \(\mathbb{R}^3\) and the equilibrium–set is a two–dimensional submanifold of \(\mathbb{R}^3\). One calls (8.30) an unfolding of the original system (8.12).

The construction of suitable unfoldings of a given parameterized system (8.26) is an important topic of bifurcation theory. We consider only another simple example involving a naturally unfolded system with a cusp bifurcation.

As a simplified, two-dimensional model of the roll-behavior of ships, consider a thin, flat lamina, that stands on edge on a plane and consists of a material, which allows the center of gravity to shift. More specifically, the lamina is assumed to have the shape of a parabola cut off by a line perpendicular to the symmetry axis as shown in Figure 8.17.

![Figure 8.17: Parabolic lamina](image1)

![Figure 8.18: Equilibrium surface](image2)

In the indicated \((\xi, \eta)\) coordinate system the parabola is given by \(\eta = \xi^2\) and \((x, y)^\top\) denotes the center of gravity. At equilibrium the line between the point of contact and the center of gravity must be perpendicular to the resting plane. In other words, if \((\xi, \xi^2)^\top\) is the point of contact, then the tangent vector \((1, 2\xi)^\top\) of the parabola at that point is in the resting plane and must be orthogonal to the vector \((x - \xi, y - \xi^2)^\top\). Hence for given \((x, y)^\top\) the coordinate \(\xi\) defining the point of contact must satisfy the equation

\[
G(\xi, x, y) := 2\xi^3 + \xi(1 - 2y) - x = 0.
\]  

(8.31)

Clearly, by 8.5.1 the set \(M\) of all solutions of (8.31) is a 2-dimensional submanifold of \(\mathbb{R}^3\). If we use a rotation, such that the base plane is spanned by the parameters \(x, y\), then Figure 8.18 shows that the manifold has a fold.

Since the potential energy of the lamina is proportional to the distance between the contact-point and the center of gravity, it follows, as in subsection 8.4.1, that the stability boundary is the curve on the manifold defined by the system of equations

\[
\begin{pmatrix}
G(\xi, x, y) \\
D_\xi G(\xi, x, y)
\end{pmatrix} := \begin{pmatrix}
\xi^3 + \xi(1 - 2y) - x \\
6\xi^2 + (1 - 2y)
\end{pmatrix} = 0.
\]  

(8.32)
The orthogonal projection of this curve onto the \((x, y)\)-plane satisfies
\[
x = -4t^3, \quad y = \frac{1}{2} + 3t^2, \quad t \in \mathbb{R},
\]
and is shown in Figure 8.19. The curve on \(M\), above the line segment marked \((1) \rightarrow (4)\), is depicted in Figure 8.20, and, on it, the points between \((3a)\) and \((2b)\) are unstable equilibria. If, the center of gravity moves from \((1)\) to \((4)\) on the line segment of Figure 8.19, then on \(M\) we reach the stability boundary at the point \((3a)\) (above \((3)\)). There we expect the point of contact to jump from \((3a)\) to the stable equilibrium \((3b)\) and then to proceed to \((4)\). Conversely, if we move from \((4)\) toward \((1)\), then the stability boundary is reached at \((2b)\) (above \((2)\)) and the point of contact will jump to the point \((2a)\) on the stable branch and continue along it to \((1)\). The sequence of points
\[
(1) \rightarrow (3a) \rightarrow (3b) \rightarrow (4) \rightarrow (2b) \rightarrow (2a) \rightarrow (1)
\]
represents a so-called \textit{hysteresis loop}.

8.6 Some Numerical Aspects

As before, consider a dynamical system (8.26) defined by a mapping \(G\) from \(\mathbb{R}^n := \mathbb{R}^m \times \mathbb{R}^d\) to \(\mathbb{R}^m\) such that \(G \in C^1(\Omega)\) for some open set \(\Omega \subset \mathbb{R}^n\). Here \(d\) is the dimension of the parameter space. For ease of notation, we assume from now on, that \(\Omega = \mathbb{R}^n\).

For the study of local bifurcations, we may concentrate on the properties of the set of equilibria
\[
M := \{(x, \mu) \in \mathbb{R}^n : G(x, \mu) = 0\}.
\]
If \(\text{rank } DG(x, \mu) = m\) for all \((x, \mu) \in M\), then \(M\) is a \(d\)-dimensional submanifold of \(\mathbb{R}^n\), and hence we want to investigate numerically the shape of this manifold.
Consider any point \((x^0, \mu^0)^\top \in M\), where \(x^0\) is a hyperbolic equilibrium and hence we know that \(\text{rank } D_x G(x^0, \mu^0) = m\). Then, by the implicit function theorem, there exist open neighborhoods \(U_d \subset \mathbb{R}^d\) and \(U_m \subset \mathbb{R}^m\) of \(\mu^0\) and \(x^0\), respectively, and a \(C^1\) mapping \(\xi : U_d \rightarrow U_m\), such that
\[
\begin{bmatrix} \xi(\mu)\end{bmatrix} = (x^0, \mu^0)^\top \quad \forall \mu \in U_d,
\]
and
\[
D_x G(\xi(\mu), \mu) D\xi(\mu) + D\mu G(\xi(\mu), \mu) = 0 \quad \forall \mu \in U_d.
\]
From this it follows that \(\varphi : U_d \rightarrow \mathbb{R}^m \times \mathbb{R}^d\), \(\varphi(\mu) = \begin{bmatrix} \xi(\mu) \end{bmatrix}^\top \quad \forall \mu \in U_d\),
is a homeomorphism of \(U_d\) onto \(M \cap (U_d \times U_m)\), and that \(D\varphi(\mu)\) is injective for \(\mu \in U_d\).

A regular, parametrized \(C^1\) curve in \(\mathbb{R}^n\) is a \(C^1\) function \(\gamma : J \rightarrow \mathbb{R}^n\) on some open interval \(J \subset \mathbb{R}^1\), such that \(D\gamma(t) \neq 0\), \(\forall t \in J\). The so-called immersion condition \(D\gamma(t) \neq 0 \quad \forall t \in J\), excludes, for instance, the occurrence of cusps, as in the case of \(\gamma : \mathbb{R}^1 \rightarrow \mathbb{R}^2\), \(\gamma(t) := (t^2, t^3)^\top\), at \(t = 0\).

A standard approach for analyzing the manifold \(M\) is to probe it along suitable curve segments, but there are also numerical methods for a direct computation of parts of the manifold itself (see, e.g., M. L. Brodzik and W. C. Rheinboldt, Comp. Math. and Appl., 28, 1994, 9-21)

Let \((x^0, \mu^0)^\top \in M\), where \(x^0\) is a hyperbolic equilibrium, and construct a local parametrization \((U_d, \varphi)\) of \(M\) near \((x^0, \mu^0)^\top\). We vary \(\mu\) only along a straight line
\[
\tau \in \mathbb{R}^1 \mapsto \mu = \mu^0 + c\tau \in \mathbb{R}^d, \quad c \in \mathbb{R}^d, \; c \neq 0.
\]
(For \(d = 1\) simply set \(c = 1\)). Since \(U_d\) is an open neighborhood of \(\mu^0\), there exists a \(\delta > 0\), such that \(\mu^0 + c\tau \in U_d\) for \(\tau\) in \(J := \{\tau : |\tau| < \delta\}\). Since \(D\gamma(\tau) = D\varphi(c\tau)c \neq 0\), it follows that
\[
\gamma : J \rightarrow M, \quad \gamma(\tau) = \varphi(c\tau) \quad \forall \tau \in J,
\]
is a regular, parametrized curve on \(M\).

The task is now to generate computational approximations of these regular, parametrized \(C^1\) curves on the equilibrium manifold \(M\). In general, however, the local parametrization \((U_d, \varphi)\) is not explicitly known; i.e., we have to work with the solution set of the equation \(G(x, \mu^0 + c\tau) = 0\) for \(x\) near \(x^0\) and \(\tau\) near zero.
For ease of notation, it is useful to consider a continuously differentiable mapping

\[ F : \Gamma \subset \mathbb{R}^m \times \mathbb{R}^1 = \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^m, \quad \Gamma \text{ open}, \quad (8.39) \]

such that

\[ \text{rank} \ DF(x, \tau) = m, \quad \forall (x, \tau) \in N := \{(x, \tau)^\top \in \Gamma : F(x, \tau) = 0\}, \quad (8.40) \]

where now the scalar parameter is denoted by \( \tau \). Thus the set \( N \) is a one-dimensional submanifold of \( \mathbb{R}^{m+1} \). In the following, we will often write \( y = (x, \tau)^\top \) for the vectors of the domain space of the mapping (8.39). Note, that in the original setting, we had \( F(x, \tau) = G(x, \mu^0 + c\tau), \quad \Gamma := \mathcal{U}_m \times \mathcal{J}, \quad \text{rank} \ DG(x^0, \mu^0) = m, \)

which implies that (8.40) holds in some neighborhood of \( (x^0, 0)^\top \).

Algorithms for approximating such a one-dimensional submanifold \( N \) of \( \mathbb{R}^{m+1} \) by a sequence of points \( y^k := (x^k, \tau^k)^\top \in \mathbb{R}^{m+1}, \ k = 0, 1, 2, \ldots \), have been a topic of long-standing in numerical mathematics and engineering. Usually, they are called numerical continuation methods, although alternate names are also in use, such as imbedding methods, homotopy methods, parameter variation methods, or incremental methods. For some references to the large literature, see, e.g., E. L. Allgower and K. Georg, Numerical Continuation Methods, Springer Verlag, 1990. We outline only briefly some of the ideas behind a typical continuation method:

Let \( y^k \in \mathbb{R}^{m+1}, \ k \geq 0, \) be a ‘current’ point of the process on or near \( N \). Then a suitable vector \( v^k \in \mathbb{R}^{m+1} \) is chosen, such that the augmented mapping

\[ \hat{F}_k(y) := \begin{pmatrix} F(y) \\ (v^k)^\top (y - y^k) \end{pmatrix} \quad (8.41) \]

has at \( y = y^k \) an invertible derivative

\[ D\hat{F}_k(y^k) = \begin{pmatrix} DF(y^k) \\ (v^k)^\top \end{pmatrix}. \quad (8.42) \]

Then, for a sufficiently small \( \eta_k \in \mathbb{R}^1 \) the equation

\[ \hat{F}_k(y) = \begin{pmatrix} 0 \\ \eta_k \end{pmatrix} \quad (8.43) \]

has a unique solution \( y = y^{k+1} \in N \), which can be computed by some iterative process, e.g., a Newton-type method, started from some appropriately chosen point near \( N \). Convergence properties of the iteration are used to control the choice of the stepsize to the next predicted point.

With this four major decisions arise in the design of this type of continuation process, namely the
(i) construction of the augmenting vectors $v^k$,
(ii) design of the iterative process and its controls,
(iii) choice of the increments $\eta_k$,
(iv) prediction of the starting point of the iteration.

A frequent choice for $v^k$ is a normalized tangent vector of $N$ at $y^k$ specified by

$$DF(y^k)u^k = 0, \quad \|u^k\|_2 = 1.$$  \hfill (8.44)

An (un-normalized) tangent vector can be obtained as solution of a linear system

$$
\begin{pmatrix}
DF(y^k) \\
v^\top
\end{pmatrix} u = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
$$

where $v$ is any vector, such that the matrix is nonsingular. The choice of a tangent vector for $v^k$ also suggests the use of a point $y^k \pm h_k v^k$ on the tangent line as start-point for the iterative process. Here the sign has to be chosen such that we follow the manifold $N$ in a specified direction. H. B. Keller in 1978 introduced the name *pseudo-arclength methods* for continuation methods based on this choice of the direction vectors.

During the continuation process we are interested in detecting bifurcation points. Let $y(s) = (x(s), \tau(s)) \in \mathbb{R}^m \times \mathbb{R}^1$ denote a smooth local parameterization of a "branch" of the manifold $N$. Then we are looking for points where $Re \lambda(s) = 0$ for at least one eigenvalue $\lambda(s)$ of $D_x F(x(s), \tau(s))$. Typically, such a point is detected by means of a test function; that is, a smooth scalar function $\psi(s) = \psi(y(s))$, which has a simple zero at bifurcation points of a particular type. Hence, such a point is found between consecutive continuation points $y(s_k)$ and $y(s_{k+1})$, when

$$\psi(s_k)\psi(s_{k+1}) < 0.$$  \hfill (8.45)

Once a bifurcation point has been detected, it is computed as the solution of some augmented system of equations. Of special interest are here minimally extended systems

$$
\begin{pmatrix}
F(x, \tau) \\
g(x, \tau)
\end{pmatrix} = 0,
$$

constructed with a suitably chosen real function $g$. But, in many cases much larger extended systems are required. Theoretically, one might use in (8.46) the test function itself; but, from the viewpoint of the required iterative process, this is often not a good choice.
Note, that the construction of a test function has to work with a smooth branch of the manifold \( N \) specified by a suitable local parametrization. On the other hand, the continuation process may well jump between disjoint parts of \( N \) and hence produce consecutive points, that are not connected by a smooth branch. Figure 8.21 shows sketches of a few possible situations, such as a jump from the neighborhood of a limit point to another part of \( N \), a jump across a gap between two neighboring branches, and – near a transcritical bifurcation – to an intersecting branch.

![Figure 8.21: Problems during a Continuation Process](image)

It should also be noted, that at a limit point or Hopf-bifurcation point \((x^*, \tau^*)^\top \in \Gamma\) we have \( \text{rank} \ DF(x^*, \tau^*) = m \). Hence \((x^*, \tau^*)^\top\) is a well-defined point of \( N \) and a well-designed continuation process should pass through such points. But at a transcritical or a cusp bifurcation the mapping \( F \) is no longer a submersion and hence we should expect computational difficulties.

These situations need to be taken into account in the design of continuation methods, in general, and of detection algorithms for bifurcation points, in particular. This cannot be discussed here; instead we consider only a few techniques for detecting and computing simple bifurcation points.

At a limit point \( y^* = (x^*, \tau)^\top \) on the one-dimensional manifold \( N \) the matrix \( D_xF(x^*, \tau^*) \) has a zero eigenvalue. Hence, if, as before, \( s \mapsto y(s) = (x(s), \tau(s))^\top \) denotes a local parametrization of a branch of \( N \), then we can use the test function

\[
\psi(s) = \det D_xF(x(s), \tau(s)) \tag{8.47}
\]

for detecting limit points by means of (8.45). However, the test function (8.47) vanishes not only at limit points, but, for example, also at a cusp bifurcation point. Therefore, a second test is needed to identify the particular bifurcation. This is not an untypical situation. In fact, in most cases, it is not possible to represent a singularity with only one test function, and one has to work with a decision-tree involving various tests for quantities to be zero or nonzero, etc.
In the case of limit points with respect to the (scalar) parameter $\tau$ another characterization derives from the fact, that at the point $y^* \in N$ the tangent space $T_{y^*}N$ of $N$ is perpendicular to the parameter space. Hence, suppose that the smooth function $s \mapsto u(s) \in T_{y(s)}N$, represents a normalized tangent vector of $N$ along our local branch. Then we can use the test function
\[ \psi(s) = e^{m+1}u(s), \quad DF(x(s), \tau(s))u(s) = 0. \] (8.48)

If a simple zero of $\psi$ has been found between two points computed by a continuation process, then one can employ very effectively some interpolation coupled with the corrector iteration of the process to evaluate the limit point (see, e.g., R. Melhem and W. Rheinboldt, Computing, 29, 1982, 201-226).

The condition (8.45) is designed to ensure that the test function has a simple zero between the two computed points. But, for example, the test function (8.48) is zero also at an inflection point, which would not agree with the normal form (8.6) of limit points. Hence, once again, there is a need for excluding such a situation, which, for limit points, leads to the use of the second derivative of $F$.

The detection of a Hopf bifurcation point $(x(s^*), \tau(s^*))^\top$ along our current branch of $N$ is more demanding. Such a point is characterized by a simple pair of purely imaginary eigenvalues $\pm i\gamma$ of $D_xF(x(s^*), \tau(s^*))$. Hence this matrix must have two distinct eigenvalues, that sum to zero, which means, that the test function
\[ \psi(s) := \prod_{j,\ell=1}^m [\lambda_j(s) + \lambda_{\ell}(s)] \] (8.49)
vanishes for $s = s^*$.

Obviously, it would be costly to evaluate the entire spectrum of $D_xF(y(s))$ at each computed point, in order to compute the test function (8.49). Here the Kronecker product of matrices can be used. Generally, the Kronecker product of any two matrices $A, B \in \mathbb{R}^{n \times n}$ is the $n^2 \times n^2$ dimensional block-matrix
\[
A \otimes B = \begin{pmatrix}
a_{11}B & a_{12}B & \cdots & a_{1n}B \\
a_{21}B & a_{22}B & \cdots & a_{2n}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1}B & a_{n2}B & \cdots & a_{nn}B
\end{pmatrix}.
\]

If $\sigma(A) = \{\lambda_1^A, \ldots, \lambda_n^A\}$ and $\sigma(B) = \{\lambda_1^B, \ldots, \lambda_n^B\}$ denote the spectra of the two matrices, then a fundamental result of C. Stephanos, 1900, states, that the eigenvalues of
\[
(I_n \otimes A) + (B \otimes I_n) = \begin{pmatrix}
A + b_{11}I_n & b_{12}I_n & \cdots & b_{1n}I_n \\
b_{21}I_n & A + b_{22}I_n & \cdots & b_{2n}I_n \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1}I_n & b_{n2}I_n & \cdots & A + b_{nn}I_n
\end{pmatrix}
\]

are all equal to $m + \lambda_j^A + \lambda_{\ell}^B$ for $j, \ell = 1, \ldots, m$. This result can be used to compute the eigenvalues of $D_xF(x(s^*), \tau(s^*))$. The spectrum of $D_xF(x(s^*), \tau(s^*))$ can then be approximated by the spectrum of $D_xF(x(s), \tau(s))$ at a nearby point, which can be computed more efficiently.
are the $n^2$ numbers $\lambda^A_j + \lambda^B_\ell$, $j, \ell = 1, \ldots, n$.

Since the determinant of any square matrix is the product of its eigenvalues, it follows, that the test function

$$
\psi(s) := \det((I_n \otimes D_x F(y(s))) + (D_x F(y(s)) \otimes I_n)),
$$

(8.50)

must vanish at a Hopf bifurcation point. While the matrix has a relatively simple form, the computation of this test function is certainly costly for larger dimensions.

Note, that (8.50) vanishes not only at Hopf points, but also at points, where two real eigenvalues of $D_x F$ sum to zero, so called neutral saddles. Thus, once again, a secondary test is needed, to distinguish the cases. In essence, this reduces to a check whether the eigenvalues, that sum to zero, have a nonzero real part or not. Here one often uses the so called Lyapunov coefficients, which represent a generalization of the eigenvalues at an equilibrium point.

For the computation of a Hopf bifurcation point $(x^*, \tau^*)^T \in N$ we require a complex eigenvector corresponding to a purely imaginary eigenvalue pair; i.e.,

$$
D_x F(x^*, \tau^*)(u \pm iv) = (\pm i\gamma)(u \pm iv) \quad u, v \in \mathbb{R}^n,
$$

or equivalently

$$
D_x F(x^*, \tau^*)u = -\gamma v, \quad D_x F(x^*, \tau^*)v = \gamma u.
$$

In order to normalize $u$ and $v$, assume that $c \in \mathbb{R}^n$ is some vector, for which $c^\top u = 0$ and $c^\top v = 1$, and consider the following augmented system of $(3n + 2)$ equations in the unknowns $x, u, v, \gamma$ and $\mu$:

$$
\begin{pmatrix}
F(x, \tau) \\
D_x F(x, \tau)u + \gamma v \\
D_x F(x, \tau)v - \gamma u \\
c^\top u \\
c^\top v - 1
\end{pmatrix} = 0.
$$

(8.51)

A. Griewank and G. W. Reddien (IMA J. Num. Anal, 3, 1983, 295-303) proved, that if $(x^*, \tau^*)^T \in N$ is a Hopf bifurcation point, then – under certain conditions – the system (8.51) has a solution $(x^*, u^*, v^*, \gamma^*, \mu^*)$ and the Jacobian of the system is nonsingular at this point. They also showed, that in the implementation of Newton’s method it is not necessary to work with the full system, but that a Newton step can be computed by means of three evaluations of the derivative $DG(x, \tau)$.

As we noted, the mapping $F$ is a submersion at a limit point or Hopf-bifurcation point $(x^*, \tau^*)^T \in \Gamma$, and hence, in principle, the continuation process passes through such points.
and their detection and computation does not cause numerical difficulties. But the situation is different at a cusp bifurcation, where $F$ is no longer a submersion. In particular, such points are sensitive to small changes of the mapping $G$ of the original multi-parameter problem (8.26). For example, the unfolded problem

$$\dot{x} = \nu + x(\mu - x^2), \quad (x, \mu, \nu)^T \in \mathbb{R}^3,$$

reduces for $\nu = 0$ to the normal form (8.12) of the cusp bifurcation, where, locally near the origin of the $(x, \mu)$-plane, the equilibrium set forms the familiar pitchfork of Figure 8.9. But, even for very small, nonzero values of $\nu$ the pitchfork disappears as Figure 8.22 shows.

![Figure 8.22: Perturbed Cusp Bifurcation](image)

Thus, numerically, we expect difficulties in detecting and computing the cusp point. In practice, it is likely, that the continuation path does not go through the point itself, but passes it only in some neighborhood. For a precise computation we should work with the unfolded problem and construct augmented systems of equations involving the mapping $G$. Here, as before, minimally extended systems are of interest, which involve only as many additional equations as the dimension $d$ of the parameter space. In other words, for $d = 2$ we want to use a system of the form

$$\begin{pmatrix} G(x, \mu) \\ g_1(x, \mu) \\ g_2(x, \mu) \end{pmatrix} = 0.$$


Larger augmented systems are also widely discussed in the literature. For example for a
scalar problem \( g(x, \mu, \nu) = 0 \), \( g : \mathbb{R}^3 \rightarrow \mathbb{R}^1 \), the system
\[
\begin{pmatrix}
g(x, \mu, \nu) \\
D_x g(x, \mu, \nu) \\
D_{xx} g(x, \mu, \nu)
\end{pmatrix} = 0
\]
can be used. For \( g(x, \mu, \nu) := \nu + x(\mu - x^2) \) this becomes
\[
\begin{pmatrix}
\nu + \mu x - x^3 \\
\mu - 3x^2 \\
-6x
\end{pmatrix} = 0,
\]
and it straightforward, that \((0, 0, 0)^\top \in \mathbb{R}^3\) is the unique solution, where the Jacobian is nonsingular and, hence, Newton’s method is locally convergent.

Suppose that for the reduced system \( F(x, \tau) = G(x, \mu^0 + c\tau) = 0 \) with the scalar parameter \( \tau \) we have computed a cusp bifurcation point \((x^*, \tau^*)^\top\), where two solution branches intersect. Then, a further task is the determination of the tangent directions of these intersecting branches. Of course, for this we must work again on the \(d\)-dimensional equilibrium manifold \( M \subset \mathbb{R}^{m+d} \) of the unfolded problem defined by \( G \).

In the mechanical example of subsection (8.4.1) the cusp bifurcation occurs at the saddle point \((0, 1, 0)^\top \in \mathbb{R}^3\) of the two-dimensional equilibrium manifold. This reflects a more general connection of these types of points with geometric properties of the equilibrium manifolds, notably its curvature behavior.

It is not feasible to discuss here the theoretical and computational aspects of the curvature of manifolds. As observed by P. J. Rabier and W. C. Rheinboldt (Numer. Math. 57, 681-694 (1990)), it is computationally advantageous to work on Riemannian manifolds with the second fundamental tensor rather than the Gaussian curvature tensor.

Suppose, that the dynamical system (8.26) now involves a mapping \( G : \Omega \subset \mathbb{R}^n := \mathbb{R}^m \times \mathbb{R}^d \rightarrow \mathbb{R}^m \), which is twice continuously differentiable on the open set \( \Omega \). Of course, we still assume, that \( \text{rank} \, DG(x, \mu) = m \) on \( M := \{(x, \mu)^\top : G(x, \mu) = 0\} \) and hence that \( M \) is a \(d\)-dimensional submanifold of \( \mathbb{R}^n \).

The tangent space \( T_y M \) of \( M \) at \( y := (x, \mu)^\top \in M \) is the \(d\)-dimensional space \( \ker \, DG(x, \mu) \) and hence the orthogonal complement \( N_y M = \ker \, DG(y)^\perp \) is an \(m\)-dimensional linear subspace of \( \mathbb{R}^n \). We may call \( N_y M \) the normal space of \( M \) at \( y \). Since \( G \) is a submersion on \( M \), the restriction \( DG(y)_{\mid N_y M} \) is a nonsingular linear mapping of \( N_y M \) onto itself. Therefore, for any \( y \in M \) the bilinear mapping
\[
V_y : T_y M \times T_y M \rightarrow N_y M \quad V_y(u, v) = -[DG(y)_{\mid N_y M}]^{-1} D^2 G(y)(u, v) \quad \forall u, v \in T_y M,
\]
(8.52)
is well defined. More specifically, \( V \) represents a symmetric, vector-valued, two-covariant tensor on \( M \) and is a characterization of the mentioned second fundamental tensor of \( M \) in terms of the submersion \( G \).

In order to connect this tensor with the curvature properties of \( M \), consider a unit tangent vector \( u \in T_y M, \|u\|_2 = 1 \) at a point \( y \in M \). Then \( z = V_y(u, u) \in N_y M \) is a normal vector of \( M \) at \( y \) and the affine plane \( y + \text{span}(u, z) \) intersects \( M \) in a curve \( \gamma \). It then follows, that \( \kappa(y) := \|z\|_2 \) is the curvature of \( \gamma \) at \( y \) and, for nonzero \( \kappa(y) \), the unit vector \( z/\|z\|_2 \) is the principal normal of \( \gamma \) at \( y \).

The representation (8.52) readily translates into a computational algorithm. Let the columns of
\[
Q \in \mathbb{R}^{n \times d}, \quad Q^\top Q = I_d, \quad \text{rge } Q = T_y M,
\]
be an orthonormal basis of \( T_y M \). Then it is readily verified, that the matrix of the linear system
\[
\begin{pmatrix}
DG(y) \\
Q^\top
\end{pmatrix} z = \begin{pmatrix}
D^2 G(y)(u, v) \\
0
\end{pmatrix}, \quad u, v \in T_y M,
\]
is nonsingular and that the (unique) solution equals \( z = V_y(u, u) \). Thus, if functions for the evaluation of the Jacobian matrix \( DG(y) \) and the vector \( D^2 G(y)(u, v) \) are available, then the computation of \( V \) requires only the solution of a linear system of dimension \( n \).

In the cited article by Rabier and Rheinboldt it is also shown how to approximate the component \( V_y(u, u) \) of \( V \) by means of the indicated geometric characterization in terms of certain curves on \( M \), without requiring the second derivative of \( G \).

Without further motivation, we consider now a special class of bifurcations on \( M \). A point \( y^* \in M \) is a foldpoint of \( M \) with respect to the parameter space \( \Pi \), if
\[
\Pi \cap N_{y^*} M \neq \{0\}. \quad (8.54)
\]
The dimension \( r := \text{dim}(\Pi \cap N_{y^*} M) > 0 \) of the subspace is called the first singularity index of \( y^* \). Thus, at \( y^* \) the directions of \( r \) parameter components are orthogonal to the tangent space. Generally, it is easily seen, that \( y^* \in M \) is a foldpoint if \( D_x G(x, \mu) \) has a zero eigenvalue. Thus, e.g., for a scalar-parameter problem limit points with respect to the parameter are foldpoints.

We fix now the \( r \) parameter components corresponding to (8.54) to their value at the foldpoint \( y^* \). This gives a reduced problem in which only the remaining \( d - r \) parameter components are variable. Then the solution set of this reduced problem in the vicinity of the foldpoint equals
\[
\{ y \in M : \langle y - y^*, z \rangle = 0 \quad \forall \quad z \in \Pi \cap N_{y^*} M \}, \quad (8.55)
\]
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and will be called the cutset of $M$ at the foldpoint $y^*$.

As an example with a two-dimensional parameter space $\Pi$, consider again the real valued equation

$$g(x, \mu, \nu) := \nu + x(\mu - x^2), \quad (x, \mu, \nu) \in \mathbb{R}^3.$$  (8.56)

At the cusp–bifurcation $y^* = (0, 0, 0)^\top \in \mathbb{R}^3$, we have $\Pi := \text{span}(e^2, e^3)$, $T_{y^*}M = \text{span}(e^1, e^2)$, and $N_{y^*}M = \text{span}(e^3)$. Thus (8.54) is satisfied and the cutset is the solution set of $x(\mu - x^2) = 0$; i.e., the familiar pitchfork.

Let $\{z^1, \ldots, z^r\}$ be an orthonormal basis of the subspace $\Pi \cap N_{y^*}M$ and extend it to an orthonormal basis $\{z^1, \ldots, z^m\}$ of all of $N_{y^*}M$. Then, under some non-degeneracy conditions (see, M. Buchner, J. Marsden and S. Schechter, J. Diff. Equ. 48, 1983,404-433, and J. P. Fink and W. C. Rheinboldt, SIAM J. Num. Anal., 24, 1987, 618-633) the form of the cutset is determined – locally near $y^*$ – by the nontrivial zeroes of the system of the $r$ quadratic equations

$$\langle V_{y^*}(u,u), z^k \rangle = 0, \quad k = 1, \ldots, r.$$  (8.57)

The solutions are the desired bifurcation directions for $M$ at the foldpoint $y^*$. We usually assume that these solutions are normed to Euclidean length one.

In the case of (8.56) a straightforward calculation shows that at $y^* = 0 \in \mathbb{R}^3$ the equations (8.57) become

$$0 = \langle V_{y^*}(u,u), e^3 \rangle = \sin(2\phi), \quad u = (\cos(\phi), \sin(\phi), 0)^\top \in T_{y^*}M.$$  (8.58)

Hence the bifurcation directions are $e^1, e^2 \in \mathbb{R}^3$; that is, exactly the direction of the intersecting branches of the pitchfork at the origin.

As a second example, consider the two–point boundary value problem

$$-u'' - \mu u + au^2 = 0, \quad u(0) = u(\pi) = 0,$$  (8.59)

where $a$ is a constant and $\mu$ a parameter. After a straightforward discretization and in unfolded form, this leads to the finite dimensional system

$$Tx + h^2(aq(x) - \mu x) + \nu w = 0, \quad x \in \mathbb{R}^m, \quad h = \frac{\pi}{m + l},$$  (8.60)

where

$$q(x) = \begin{pmatrix} x_1^2 \\ \vdots \\ x_m^2 \end{pmatrix}, \quad T = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}.$$  157
Let $\lambda_1 < \ldots < \lambda_m$ be the eigenvalues of $T$ and $w$ a normalized eigenvector of $T$ corresponding to the eigenvalue $\lambda_k$ for a given $k$. Then $y^* = (0, \lambda_k/k^2, 0)$ turns out to be a foldpoint of (8.60) with respect to the space of the parameters $\mu$ and $\nu$. Moreover, a closer analysis shows that for odd values of $k$ the bifurcation is transcritical, while for even $k$ the branches intersect at a right angle.

The system of quadratic equations (8.57) is readily solvable. If \{\(u^1, \ldots, u^d\)\} is a basis of $T_{y^*}M$ for which the components $V_{ij} = V_{y^*}(u^i, u^j)$, $1 \leq i \leq j \leq d$, of the second fundamental tensor have been computed, then (8.57) reduces to a system of homogeneous quadratic equations

$$\xi^T A_k \xi = 0, \quad \xi \in \mathbb{R}^d, \; \xi \neq 0, \quad u = \xi_1 u^1 + \ldots + \xi_d u^d,$$

(8.61)

where the symmetric $d \times d$ matrices $A_k$ have the elements

$$a^k_{ij} = \langle V_{ij}, z^k \rangle, \quad 1 \leq i \leq j \leq d, \; k = 1, \ldots, r.$$

This system of quadratic equations can have finitely many isolated solutions only in the case $d = r + 1$. The case $d = 2$, $(r = 1)$ is trivial, and also for $d = 3$, $(r = 2)$ simple methods are available. For larger dimensions there exists software for the numerical of systems of polynomial equations, which can be applied here.