Exercise Sheet 6

Exercise 19 (Parareal method)

Consider the Matlab-implementations `parareal_1d_exp.m` and `parareal_1d_hyp.m` on the website, which (attempt to) solve the ODEs

$$u' = \lambda u$$

in the interval $(0, 1)$ with initial value $u(0) = 1$ and

$$u' = u^2$$

in the interval $(0, T)$ with initial value $u(0) = u_0$, respectively, by the Parareal method where the propagators $\mathcal{F}$ and $\mathcal{G}$ have been chosen as the implicit Euler method with different step sizes. (For simplicity, in the notations of the lecture: $\delta t < \delta T = \Delta T$.)

a) The number of coarse and fine sweeps is K and $K - 1$, respectively. For fixed $K > 1$, the algorithm can be interpreted as new time-integration scheme. Study the order of the discretization error of this method w. r. t. the coarse step size ΔT for $K = 2$ and $K = 3$.

b) Extend `parareal_1d_exp.m` to systems of linear ODEs

$$\boldsymbol{u}' = \boldsymbol{A}\boldsymbol{u}$$

with $\boldsymbol{A} \in \mathbb{R}^d$, $d \in \mathbb{N}$, given.

Exercise 20 (Schwarz methods: miscellaneous)

a) Prove Lemma 4.1 of the lecture.

b) Let $V \subset L^2(\Omega)$ be finite dimensional with basis $(\varphi_p)_{p\in\mathcal{N}}$. For a function $r \in V$, we consider the corresponding functional $(r, \cdot)_{L^2(\Omega)} \in V'$. Let $\boldsymbol{R} := \big((r, \varphi_p)_{L^2(\Omega)}\big)_{p\in\mathcal{N}}$. Consider a subspace $V_i \subset V$ with basis $(\varphi_p^i)_{p\in\mathcal{N}_i}$ where the natural embedding w. r. t. the chosen bases is represented by the matrix $\boldsymbol{I}_i \in \mathbb{R}^{|\mathcal{N}|\times|\mathcal{N}_i|}$.

Show that $\boldsymbol{R}_i := \big((r, \varphi_p)_{L^2(\Omega)}\big)_{p\in\mathcal{N}_i} = \boldsymbol{I}_i^T \boldsymbol{R}$. (See also Exercise 9.)

c) Derive expressions for the right hand side of the preconditioned system (30) of the lecture both in operator form and in matrix form.

d) Show that, under the assumptions of Lemma 4.2 of the lecture, the subspace solution operators $T_i : V \to V$ are projections, i. e., $T_i^2 = T_i$.

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Exercise 21 (Hybrid Schwarz methods)

We proceed in two steps to derive hybrid realizations of the abstract Schwarz method (i. e., algorithms that apply the subspace corrections partly additively and partly multiplicatively). For simplicity, only subspaces are considered here. Assume that the space V is first split into

$$V = V_0 + V_*$$

and then V_* is split into

$$V_* = V_1 + \ldots + V_K.$$

Analogously to Example 4.3 and Example 4.4 of the lecture, calculate an expression in the subspace solution operators $\{T_i\}_{i=0,\ldots,K}$ for the preconditioned operator P_{hybrid} corresponding to the Schwarz method that is

a) multiplicative w. r. t. the first splitting and additive w. r. t. the second splitting,

b) additive w. r. t. the first splitting and multiplicative w. r. t. the second splitting.

Remark: The condition number of P_{hybrid} is expected to lie between the ones of the purely additive and the purely multiplicative variants P_{ad}, P_{mu}.

Exercise 22 (Implementation of the two-level additive Schwarz method)

Implement the two-level additive Schwarz method with exact subdomain solvers and variable overlap for the 1d model problem.

For simplicity, you may assume

I) that $|\mathcal{T}_h|$ is odd and that $|\mathcal{N}_h| = |\mathcal{T}_h| - 1$ is divisible by the number of subdomains K (this facilitates an equal distribution of the mesh with minimal overlap) and

II) that $|\mathcal{T}_h|$ is divisible by $|\mathcal{T}_H|$ (this facilitates a nested coarse mesh).

For example, structure your code in Matlab as follows:

a) Write a main function that sets up the problem and the ingredients for the Schwarz preconditioner.

b) Implement a routine that applies the (symmetric) preconditioner to a given vector (namely, to the residual).

c) Use this routine (as function handle) in the `pcg` of Matlab.