Exercise Sheet 5

Exercise 16 (Uzawa's algorithm)

Consider the saddle point problem

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^T & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{pmatrix} \qquad (\star)$$

with $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ symmetric positive definite and $\boldsymbol{B} \in \mathbb{R}^{n\times m}$. Show that, for given $\boldsymbol{\lambda}^0 \in \mathbb{R}^m$ and damping parameter $\omega > 0$, Uzawa's algorithm

For $j = 1, 2, \ldots$ {

$$\begin{aligned} \text{Solve} \quad \boldsymbol{A}\boldsymbol{u}^j &= \boldsymbol{f} - \boldsymbol{B}\boldsymbol{\lambda}^{j-1} \\ \text{Update} \quad \boldsymbol{\lambda}^j &= \boldsymbol{\lambda}^{j-1} + \omega\left(\boldsymbol{B}^T\boldsymbol{u}^j - \boldsymbol{g}\right) \end{aligned}$$

}

is equivalent to a Richardson iteration (without preconditioner) for the Schur complement system (i. e., the reduced equation in $\boldsymbol{\lambda}$) associated with $(\star)$.

Exercise 17 (Parallel implementation of the Dirichlet-Neumann method)

Develop a concept (pencil and paper) for a parallel implementation of the Dirichlet-Neumann algorithm with many subdomains in a distributed memory environment. In particular, address the question how to initially distribute the data and when to communicate between the processors. What are reasonable choices for the number of subdomains, number of processors, number of degrees of freedom? Special care should be spent on the solution of the coarse problem arising from the vertex Schur complement as discussed in the lecture. See also the Matlab-file

`dirichletneumann_2d_modelproblem_manysubdomains.m`

on the website.

Exercise 18 (Discretization error of the mortar method)

In this exercise, we develop step by step an a priori estimate for the mortar method (cf. Thm. 2.38 of the lecture).

For $i = 1, 2$, let $V_{i,h} \subset H^1(\Omega_i)$ be the local finite element spaces (with orders r_i and mesh sizes h_i) satisfying zero Dirichlet boundary conditions on $\partial\Omega$. (Assume that the $(d-1)$-dimensional measure of $\partial\Omega \cap \partial\Omega_i$ is positive for $i = 1, 2$.) The corresponding discrete trace spaces on the interface $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$ are $\Lambda_{i,h}$, $i = 1, 2$.

For a function in the product space $v_h = \left(v_h^{(1)}, v_h^{(2)}\right) \in X_h := V_{1,h} \times V_{2,h}$, the jump across Γ is denoted by

$$[v_h] := v_h^{(1)}\Big|_\Gamma - v_h^{(2)}\Big|_\Gamma \in L^2(\Gamma).$$

Then, the non-conforming approximation space of the mortar method is defined by

$$V_h := \left\{ v_h \in X_h \;\middle|\; \left([v_h], \mu_h^{(2)}\right)_{L^2(\Gamma)} = 0 \;\; \forall\, \mu_h^{(2)} \in \Lambda_{2,h} \right\} \not\subset H_0^1(\Omega). \qquad (\star)$$

For a function $v \in L^2(\Omega)$ with $v|_{\Omega_i} \in H^1(\Omega_i)$, $i = 1, 2$, we use the norm

$$\|v\|_* := \left(\sum_{i=1}^{2} \|v\|_{H^1(\Omega_i)}^2 \right)^{1/2}$$

and the semi-norm $|v|_*$ analogously. Note that a Poincaré inequality

$$|v_h|_* \geq \alpha \|v_h\|_* \quad \forall\, v_h \in V_h$$

holds with a constant $\alpha > 0$ thanks to the boundary conditions.

Let $u \in H_0^1(\Omega)$ be the exact solution and $u_h \in V_h$ be the solution of the respective discrete variational problem

$$a(u_h, v_h) = \ell(v_h) \quad \forall\, v_h \in V_h.$$

a) Show that, for all $v_h \in V_h$,

$$\alpha^2 \|u_h - v_h\|_*^2 \leq \|u - v_h\|_* \|u_h - v_h\|_* + \left| \left(\frac{\partial u}{\partial n}, [u_h - v_h] \right)_{L^2(\Gamma)} \right|.$$

(Use Green's first identity assuming sufficient regularity.)

b) Conclude that

$$\|u - u_h\|_* \leq \left(1 + \alpha^{-2}\right) \inf_{v_h \in V_h} \|u - v_h\|_* + \alpha^{-2} \sup_{w_h \in V_h} \frac{\left| \left(\frac{\partial u}{\partial n}, [w_h]\right)_{L^2(\Gamma)} \right|}{\|w_h\|_*}.$$

This means that (similarly to Strang's lemma known from your finite element course) the discretization error is bounded by the best approximation error and a term accounting for the non-conformity.

Note that by ($\star$) we have

$$\left(\frac{\partial u}{\partial n}, [w_h]\right)_{L^2(\Gamma)} = \left(\frac{\partial u}{\partial n} - \mu_h^{(2)}, [w_h]\right)_{L^2(\Gamma)}$$

for any $w_h \in V_h$ and $\mu_h^{(2)} \in \Lambda_{2,h}$.

c) Show that there is a constant $C_1 > 0$ such that

$$\sup_{w_h \in V_h} \frac{\left|\left(\frac{\partial u}{\partial n}, [w_h]\right)_{L^2(\Gamma)}\right|}{\|w_h\|_*} \le C_1 h_2^{r_2} \|u\|_{H^{r_2+1}(\Omega_2)}.$$

For this purpose, use (I) the fact that there is a function $\lambda_h^{(2)}$ (e. g., the finite element interpolant of $\frac{\partial u}{\partial n}|_\Gamma$ in the space $\Lambda_{2,h}$) that satisfies

$$\left\|\frac{\partial u}{\partial n} - \lambda_h^{(2)}\right\|_{L^2(\Gamma)} \le C_2 h_2^{r_2 - 1/2} \left\|\frac{\partial u}{\partial n}\right\|_{H^{r_2-1/2}(\Gamma)}$$

and (II) the following trace inequality

$$\left\|\frac{\partial u}{\partial n}\right\|_{H^{r_2-1/2}(\Gamma)} \le C_3 \|u\|_{H^{r_2+1}(\Omega_2)}$$

provided $u|_{\Omega_2}$ is sufficiently smooth. Further, exploit (III) that for a function satisfying the mortar constraints, i. e., $w_h = \left(w_h^{(1)}, w_h^{(2)}\right) \in V_h$, we have that $w_h^{(2)}$ is the orthogonal projection of $w_h^{(1)}$ to $\Lambda_{2,h}$ w. r. t. the inner product $(\cdot,\cdot)_{L^2(\Gamma)}$; thus,

$$\left\|w_h^{(1)} - w_h^{(2)}\right\|_{L(\Gamma)} \le C_4 h_2^{1/2} \left\|w_h^{(1)}\right\|_{H^{1/2}(\Gamma)}.$$

Combined with an a priori estimate for the best approximation in V_h (not considered here), this yields

$$\|u - u_h\|_* \le C \left(\sum_{i=1}^{2} h_i^{r_i} \|u\|_{H^{r_i+1}(\Omega_i)}\right)$$

provided $u|_{\Omega_i}$, $i = 1, 2$ is sufficiently regular.