Exercise Sheet 4

Please remember to solve Exercise 12 of the previous sheet.

Exercise 13 (Implementation of the Dirichlet-Neumann method)

a) Derive a matrix representation of the Dirichlet-Neumann method using the notations of Section 2.4 of the lecture. (This means: Write down the respective linear systems that need to be solved in the two substeps without referring to the Schur complement matrix.)

b) Implement the algorithm for the 2d model problem with a simple decomposition, e. g., as in Exercise 12.

c) Visualize the iterates.

Exercise 14 (Preconditioners based on the Schur complement matrix)

Let

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^T & \boldsymbol{C} \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}$$

with $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{C} \in \mathbb{R}^{m\times m}$ symmetric positive definite and $\boldsymbol{B} \in \mathbb{R}^{n\times m}$. Consider the block decomposition

$$\boldsymbol{M} = \boldsymbol{L}\boldsymbol{U} := \begin{pmatrix} \boldsymbol{I} & 0 \\ \boldsymbol{B}^T\boldsymbol{A}^{-1} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ 0 & \boldsymbol{S} \end{pmatrix}$$

with the Schur complement matrix $\boldsymbol{S} = \boldsymbol{C} - \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B} \in \mathbb{R}^{m\times m}$. Let now $\boldsymbol{P} \in \mathbb{R}^{m\times m}$ be a preconditioner (i. e., a symmetric positive definite approximation) of $\boldsymbol{S}$. This induces a preconditioner $\boldsymbol{Q} \in \mathbb{R}^{(n+m)\times(n+m)}$ of $\boldsymbol{M}$ via

$$\boldsymbol{Q} = \boldsymbol{L}\widetilde{\boldsymbol{U}} := \boldsymbol{L}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ 0 & \boldsymbol{P} \end{pmatrix}. \tag{$\star$}$$

Show that an eigenvalue $\lambda \neq 1$ of $\boldsymbol{Q}^{-1}\boldsymbol{M}$ is also an eigenvalue of $\boldsymbol{P}^{-1}\boldsymbol{S}$.

Remark

In the context of the lecture, the system $\boldsymbol{M}$ arises from a decomposed finite element discretization and $\boldsymbol{A}$ and $\boldsymbol{B}$ have again block structure; in particular, $n = |\mathcal{N}_{h,1} \cup \mathcal{N}_{h,2}|$ and $m = |\mathcal{N}_{h,\Gamma}|$. The assertion of the exercise then means: An optimal† preconditioner $\boldsymbol{P}$ of $\boldsymbol{S}$ induces an optimal preconditioner $\boldsymbol{Q}$ of $\boldsymbol{M}$ via ($\star$).

† $\kappa(\boldsymbol{P}^{-1}\boldsymbol{S})$ is bounded uniformly w. r. t. the mesh size h

Exercise 15 (Questions)

Prepare three reasonable questions regarding Section 2 “Non-overlapping domain decomposition methods” of the lecture. (You do not need to be able to answer them.)