Exercise Sheet 3

Exercise 9 (L^2-projection)

a) Let $\Omega \subset \mathbb{R}^n$ be a domain and $V_h \subset C^0(\Omega)$ a finite element space with nodal basis $(\lambda_p^h)_{p\in\mathcal{N}_h}$. We consider the L^2-orthogonal projection $Q_h : L^2(\Omega) \to V_h$ defined by the variational equation

$$u \mapsto Q_h u : \; (Q_h u, v)_{L^2(\Omega)} = (u, v)_{L^2(\Omega)} \;\; \forall\, v \in V_h.$$

For given $u \in L^2(\Omega)$, determine the linear system in $\mathbb{R}^{|\mathcal{N}_h|}$ that needs to be solved to obtain $Q_h u$.

b) Let now $V_{h'} \subset C^0(\Omega)$ be a second finite element space. Derive a matrix representation of the restricted operator $Q_h : V_{h'} \to V_h$.

c) What can you say about the support of a function $Q_h \lambda_p^{h'}$, $p \in \mathcal{N}_{h'}$?

d) What happens in b) and c) if $V_{h'} \subset V_h$?

Remark

In the context of the lecture, the above situation occurs, e. g., when information needs to be transferred across an interface (for instance, in the Dirichlet-Neumann method). If the finite element spaces of the two neighboring subdomains do not match (i. e., the respective trace spaces are not equal), an approximation operator in the fashion of Q_h may be employed. Such a non-matching situation may be due to the use of non-matching/independent meshes in the subdomains or to different polynomial degrees.

Additionally

e) Can you estimate the condition number of the linear system in part a), possibly under additional assumptions on the finite element space?

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Exercise 10 (Sobolev spaces of fractional order)

As introduced in the lecture, Sobolev spaces with real exponents appear naturally when studying traces. This exercise further illustrates their properties.

Task

Let $\Omega = (-1, 1)$. Show that

$$u : \Omega \to \mathbb{R}, \quad u(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

satisfies $u \in H^s(\Omega)$ for $0 \leq s < 1/2$.

Remark

One can show that $v : \Omega \to \mathbb{R}$ with $v(x) = \sqrt{x}$ satisfies $v \in H^{1/2}(\Omega)$ but $v \notin H^1(\Omega)$.

Additional challenge (!)

Let Ω be a domain of your choice, e. g., $\Omega = \mathbb{R}_+$. Find a function $w \in H^{1/2}(\Omega)$ such that the weak derivative w' satisfies $w' \in \left(H_{00}^{1/2}(\Omega)\right)'$ but $w' \notin \left(H_0^{1/2}(\Omega)\right)' = \left(H^{1/2}(\Omega)\right)'$.

Exercise 11 (Properties of the Steklov-Poincaré operator)

Prove Theorem 2.24 of the lecture (i. e., continuity, ellipticity and symmetry of the Steklov-Poincaré operators $S : \Lambda \to \Lambda'$ and $S_i : \Lambda \to \Lambda'$ for $i = 1, 2$) and consider the implications.

Exercise 12 (Condition number of the Schur complement system)

Consider the model problem (Example II of the first lecture: discretization of the Poisson problem on the square $\Omega = (0, 1)^2$ with first-order Lagrange finite elements on a regular triangular mesh with $h = \frac{1}{n+1}$ and $N = n^2$ interior nodes). *You may assume that n is odd.* The stiffness matrix of this model problem can be found e. g., in the Matlab file `blockjacobi_2d_modelproblem.m` on the website.

Let the computational domain be decomposed into

$$\Omega_1 = (0, 1) \times (0.5, 1) \text{ and } \Omega_2 = (0, 1) \times (0, 0.5)$$

such that $\Gamma = (0, 1) \times \{0.5\}$ is the interface.

a) Assemble the Schur complement matrix associated with this decomposition.

b) Compare the condition numbers of the original (sparse) system and the (dense) Schur complement system: Plot these quantities for $n = 2^\ell - 1$ with $\ell = 2, \ldots, 8$ and estimate the orders w. r. t. the mesh size h.