Exercise Sheet 2

Exercise 5 (Robin-Robin method)

In the lecture, we considered the Dirichlet-Neumann method as an example for a non-overlapping domain decomposition method and studied its convergence for the 1D toy problem; see also the Matlab file `dirichletneumann_1d_parabola.m` on the website.

In this exercise (same setting as in Section 1.4 of the lecture), the transmission conditions are modified as follows. On the artificial boundary (i. e., the interface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$), linear combinations of function values and normal derivatives (co-called Robin boundary conditions) are prescribed. Choose a weight $\theta > 0$. Then, each step $(j-1) \mapsto j$ of the iteration consists of the following two parts.

$$\text{Find } u_1^j : \Omega_1 \to \mathbb{R} \text{ such that } \begin{cases} -\Delta u_1^j &= f & \text{in } \Omega_1, \\ u_1^j &= 0 & \text{on } \partial\Omega \cap \partial\Omega_1, \\ \dfrac{\partial u_1^j}{\partial n} + \theta u_1^j &= \dfrac{\partial u_2^{j-1}}{\partial n} + \theta u_2^{j-1} & \text{on } \Gamma. \end{cases}$$

$$\text{Find } u_2^j : \Omega_2 \to \mathbb{R} \text{ such that } \begin{cases} -\Delta u_2^j &= f & \text{in } \Omega_2, \\ u_2^j &= 0 & \text{on } \partial\Omega \cap \partial\Omega_2, \\ \dfrac{\partial u_2^j}{\partial n} - \theta u_2^j &= \dfrac{\partial u_1^j}{\partial n} - \theta u_1^j & \text{on } \Gamma. \end{cases}$$

a) Show that the limit $(u_1^\star, u_2^\star)$, assuming that it exists, satisfies the standard transmission conditions known from the lecture, namely

$$\begin{cases} u_1^\star &= u_2^\star & \text{on } \Gamma, \\ \dfrac{\partial u_1^\star}{\partial n} &= \dfrac{\partial u_2^\star}{\partial n} & \text{on } \Gamma. \end{cases}$$

b) Implement the algorithm (which is called Robin-Robin method) for the 1D toy problem. For this purpose, you may modify the given implementation of the Dirichlet-Neumann algorithm: Implement a new function

```
function rho = robinrobin_1d_parabola(theta,L,N,seq,plotiterates)
```

reusing the structure of the given one, `dirichletneumann_1d_parabola(...)`.

Exercise 6 (Equivalence of the weak multi-domain formulation)

Consider the standard weak formulation of the Poisson problem with the usual notations:

Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v)_{L^2(\Omega)} \quad \forall\, v \in H_0^1(\Omega). \tag{$\star$}$$

Given the (non-overlapping) decomposition $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Gamma := \partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$, let the restricted bilinear forms be defined by

$$a_i : H^1(\Omega_i) \times H^1(\Omega_i) \to \mathbb{R}, \quad a_i(v_i, w_i) := (\nabla v_i, \nabla w_i)_{L^2(\Omega_i)} := \int_{\Omega_i} \nabla v_i \cdot \nabla w_i \, \mathrm{d}\boldsymbol{x}, \quad i = 1, 2.$$

Assume that for the trace space on the interface

$$\Lambda := \{\lambda : \Gamma \to \mathbb{R} \mid \exists\, v \in H_0^1(\Omega) \text{ such that } v|_\Gamma = \lambda\},$$

we have extension operators

$$\mathcal{E}_i : \Lambda \to \{v_i \in H^1(\Omega_i) \mid v|_{\partial\Omega \cap \partial\Omega_i} = 0\}, \quad i = 1, 2,$$

satisfying $(\mathcal{E}_i \lambda)|_\Gamma = \lambda$ for all $\lambda \in \Lambda$, $i = 1, 2$. Now, consider the following multi-domain formulation of the Poisson problem:

Find $u_i \in \{v_i \in H^1(\Omega_i) \mid v|_{\partial\Omega \cap \partial\Omega_i} = 0\}$, $i = 1, 2$, such that

$$\left\{ \begin{array}{rcll} a_1(u_1, v_1) & = & (f, v_1)_{L^2(\Omega_1)} & \forall\, v_1 \in H_0^1(\Omega_1), \\ u_1 & = & u_2 & \text{on } \Gamma, \\ a_2(u_2, v_2) & = & (f, v_2)_{L^2(\Omega_2)} & \forall\, v_2 \in H_0^1(\Omega_2), \\ a_1(u_1, \mathcal{E}_1\lambda) + a_2(u_2, \mathcal{E}_2\lambda) & = & (f, \mathcal{E}_1\lambda)_{L^2(\Omega_1)} + (f, \mathcal{E}_2\lambda)_{L^2(\Omega_2)} & \forall\, \lambda \in \Lambda. \end{array} \right. \tag{$\star\star$}$$

Show:

a) If u solves ($\star$), then the pair $(u_i := u|_{\Omega_i})_{i=1,2}$ solves ($\star\star$).

b) If the pair $(u_i)_{i=1,2}$ solves ($\star\star$), then

$$u := \begin{cases} u_1 & \text{in } \Omega_1, \\ u_2 & \text{in } \Omega_2 \end{cases}$$

solves ($\star$).

Remark

One purpose, among others, of this exercise and the next one is to recap the weak formulation of elliptic PDEs.

Exercise 7 (Continuous piecewise smooth functions are weakly differentiable)

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a triangulation $\mathcal{T}$ (i. e., $\overline{\Omega} = \bigcup_{T \in \mathcal{T}} \overline{T}$). The function $u : \Omega \to \mathbb{R}$ may be piecewise continuously differentiable w. r. t. $\mathcal{T}$ (i. e., $u|_T \in \mathcal{C}^1(T)$ for all triangles $T \in \mathcal{T}$). Show that

$$u \in H^1(\Omega) \quad \Longleftrightarrow \quad u \in \mathcal{C}^0(\overline{\Omega}).$$

Exercise 8 (Performance models)

We study a very simple performance model for the computation of the residual norm on a system of $p > 0$ processors.

Let a matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be given, $n > 0$ divisible by $p = 2^\ell$ for some $\ell \in \mathbb{N}$. Assume that each processor knows n/p rows of the matrix $\boldsymbol{A}$ (i. e., processor number k knows the rows $i = (k-1)n/p, \ldots, kn/p$) and the full vectors $\boldsymbol{x}, \boldsymbol{b} \in \mathbb{R}^n$.

Assume that all computation steps (multiplication, addition, square root. . .) and all communication steps take one time unit each. Further, a processor can only send or receive one message at a time.

Calculate the time required to compute the quantity $\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|$ if

a) all processors are interconnected ("completely-connected network"),

b) one processor is interconnected to all processors, no other interconnections ("star-connected network").