Exercise Sheet 1

Exercise 1 (Solvers are preconditioners – preconditioners can be solvers)

In this exercise, we study the connections between preconditioners and stationary iterative methods. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{N\times N}$ be symmetric positive definite. For the equation $\boldsymbol{Ax} = \boldsymbol{b}$ in $\mathbb{R}^N$, consider the linear iteration

$$\boldsymbol{x}^{j+1} = \boldsymbol{x}^j + \omega \boldsymbol{B}^{-1}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}^j). \qquad (\star)$$

a) Let $\omega = 1$. Assume that the spectral radius of the corresponding iteration matrix $\boldsymbol{G} = \boldsymbol{I} - \boldsymbol{B}^{-1}\boldsymbol{A}$ satisfies $\varrho := \lambda_{\max}(\boldsymbol{G}) = \|\boldsymbol{G}\| := \max_{\|\boldsymbol{v}\|=1}\langle \boldsymbol{Gv}, \boldsymbol{v}\rangle < 1$. Show that the spectral condition number of $\boldsymbol{B}^{-1}\boldsymbol{A}$ satisfies

$$\kappa(\boldsymbol{B}^{-1}\boldsymbol{A}) := \frac{\lambda_{\max}(\boldsymbol{B}^{-1}\boldsymbol{A})}{\lambda_{\min}(\boldsymbol{B}^{-1}\boldsymbol{A})} \leq \frac{1+\varrho}{1-\varrho}.$$

Hints

i) Show that

$$\frac{|\langle \boldsymbol{Av}, \boldsymbol{v}\rangle - \langle \boldsymbol{AB}^{-1}\boldsymbol{Av}, \boldsymbol{v}\rangle|}{\langle \boldsymbol{Av}, \boldsymbol{v}\rangle} \leq \varrho \quad \forall\, \boldsymbol{v} \in \mathbb{R}^N.$$

ii) Show that the estimate

$$\mu_0\langle \boldsymbol{Av}, \boldsymbol{v}\rangle \leq \langle \boldsymbol{AB}^{-1}\boldsymbol{Av}, \boldsymbol{v}\rangle \leq \mu_1\langle \boldsymbol{Av}, \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \mathbb{R}^N$$

implies that $\kappa(\boldsymbol{B}^{-1}\boldsymbol{A}) \leq \frac{\mu_1}{\mu_0}$.

b) Show that the iteration $(\star)$ converges for $\omega \in (0, 2/\lambda_{\max}(\boldsymbol{B}^{-1}\boldsymbol{A}))$.

Remarks

Part a) implies that the preconditioned conjugate gradient (pcg) method is always faster because

$$\varrho_{\mathrm{pcg}} = \frac{\sqrt{\kappa(\boldsymbol{B}^{-1}\boldsymbol{A})} - 1}{\sqrt{\kappa(\boldsymbol{B}^{-1}\boldsymbol{A})} + 1} = \frac{1 - \sqrt{1-\varrho^2}}{\varrho} < \varrho.$$

Part b) means that a good preconditioner does not necessarily converge when used as "stand-alone solver" without damping.

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Exercise 2 (Jacobi and Gauß-Seidel as subspace correction methods)

Let V be a finite element space and let

$$V \ni v \mapsto \mathcal{J}(v) := \frac{1}{2}a(v,v) - f(v)$$

be the quadratic energy functional corresponding to the variational formulation in V with the bilinear form $a(\cdot,\cdot)$ and the right hand side $f(\cdot)$. Given a decomposition/splitting of the space $V = V_1 + V_2 + \ldots + V_m$, consider the following two algorithms.

Algorithm SSC – Sequential subspace correction method
{
Let $u_0 \in V$ be given.
For $k = 1, 2, \ldots, m$ {
Compute $v_k \in V_k$ such that $\mathcal{J}(u_{k-1}+v_k) \leq \mathcal{J}(u_{k-1}+v) \quad \forall\, v \in V_k$
Update $u_k = u_{k-1} + v_k$
}
Return value is $u_m \in V$.
}

Algorithm PSC – Parallel subspace correction method
{
Let $u_0 \in V$ be given.
For $k = 1, 2, \ldots, m$ {
Compute $v_k \in V_k$ such that $\mathcal{J}(u_0 + v_k) \leq \mathcal{J}(u_0 + v) \quad \forall\, v \in V_k$
}
Update $u_1 = u_0 + \sum_{k=1}^{m} v_k$
Return value is $u_1 \in V$.
}

Let now $m = N = \dim V$ and $V_k = \mathrm{span}(\lambda_k)$, $k = 1, \ldots, N$. (This means $\dim V_k = 1$ for $k = 1, \ldots, N$.) Show that in this case

a) Algorithm SSC is equivalent to one step of the Gauß-Seidel method and

b) Algorithm PSC is equivalent to one step of the Jacobi method.

Exercise 3 (**Matrix-free red-black Gauß-Seidel method**)

Consider the model problem (Example II of the first lecture: discretization of the Poisson problem on the square $\Omega = (0,1)^2$ with first-order Lagrange finite elements on a regular triangular mesh with $h = \frac{1}{n+1}$ and $N = n^2$ interior nodes). Implement

(a) the standard/lexicographic Gauß-Seidel method and

(b) the red-black Gauß-Seidel method

without assembling the stiffness matrix.

Hints

You may

- enlarge the set of degrees of freedom by including the $4(n+1)$ boundary nodes,
- index the degrees of freedom by double indices $(i,j)_{i=1,\dots,n+2;j=1,\dots,n+2}$, i. e., store all vectors as $(n+2)\times(n+2)$-matrices.

Exercise 4 (**Counterexample for a trace theorem**)

As indicated in the last exercise lesson, artificially introduced interfaces and function values on them play a major role in domain decomposition methods.

Trace theorem. For a domain Ω with Lipschitz boundary that satisfies an interior cone condition, a unique continuous linear mapping

$$\gamma : H^1(\Omega) \to L^2(\partial\Omega)$$

(the trace operator) exists such that

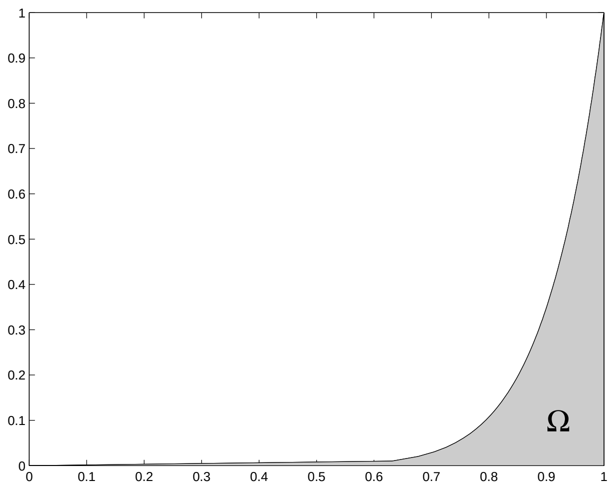
$$\gamma(v) = v|_{\partial\Omega} \quad \forall\, v \in H^1(\Omega) \cap \mathcal{C}^0(\overline{\Omega}).$$

We will later need refined versions of this theorem. We now consider a classical counter-example that illustrates what can go wrong, if the assumptions are violated.

Let $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1,\, 0 < y < x^5\}$ and $\Gamma = (0, 1) \times \{0\}$. Show that

$$u : \Omega \to \mathbb{R}, \quad u(x, y) := \frac{1}{x}$$

satisfies $u \in H^1(\Omega)$ but $u|_\Gamma \notin L^2(\Gamma)$.



Domain Ω that violates the interior cone condition

Remark

You can solve this exercise even if you do not feel very comfortable with Sobolev spaces. Just calculate the integrals $\|u\|^2_{H^1(\Omega)}$, $\|u\|^2_{L^2(\Gamma)}$ (exploiting the fact that the strong and weak derivatives coincide whenever they are both defined).