Analysis of a bacterial model with nutrient-dependent degenerate diffusion

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We introduce and study a degenerate reaction-diffusion system which can serve as a model prototype for the pattern formation of a bacterial multicellular community where the bacteria produce biofilm, grow and spread in the presence of a nutrient. Under proper conditions on the reaction terms, we prove the global existence and the uniqueness of solutions and illustrate the possible model behaviour in numerical simulations for a two-dimensional setting. Copyright © 20XX John Wiley & Sons, Ltd.

Keywords: biofilms; degenerate diffusion; parabolic system; pattern formation; weak solution; well-posedness

1. Introduction

In this article, we consider a model which can be used to describe the pattern formation of a bacterial multicellular community. A typical example for such a bacterium is the \textit{Bacillus subtilis}. In experiments, these bacteria can be brought onto agar plates containing nutrients where they grow in communities producing biofilm (the microorganisms are embedded in a self-produced matrix of extracellular polymeric substances) and thereby form patterns. Many factors influence the bacterial growth and spreading mechanism, thus resulting in different pattern formation. Among those factors are: the concrete type of a bacterial strain or mutant, the nutrient availability and the hardness of the underlying agar plate. For example, unfavourable ambient conditions, such as low nutrient levels, may cause the bacteria to switch from their active state into an inactive state, also called quiescent. Being quiescent means that the cells hardly (or not at all) move by their own but they can survive for a long period of time without nutrients \cite{1, 2}.

In our present study, we restrict ourselves to the more relevant active bacteria. We construct a model that can capture rather general bacterial growth and pattern formation depending on the available nutrients which are introduced into agar in a Petri dish and then are consumed by the bacteria. Our goal here is not to find the most realistic model description tailored to a concrete experiment, but, rather, an abstract model framework, and to analyse its main properties. Some more specific models for this bacterium can be found in \cite{3, 4}.

The paper is organised as follows. Section 2 begins with a brief preliminary Subsection 2.1 which contains some notation that we use throughout the paper. In Subsection 2.2, we introduce our mathematical model - the system (2.1) - and the model assumptions, and state the main result of the paper - Theorem 2.6. Section 2 ends with Subsection 2.3 where we compare our model to a system with a porous medium-type degeneracy. In Section 3, we first derive a local existence result for system (2.1) (Lemma 3.1) and use it afterwards to prove the existence part of Theorem 2.6. Local and global a priori estimates play the key role in both

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Section 4 is devoted to the proof of the uniqueness part of Theorem 2.6. Finally, in Section 5, we illustrate the possible model behaviour in numerical simulations in the two-dimensional case.

2. Problem setting and main result

2.1. Notation and functional spaces

We assume the reader to be familiar with the standard $L^p$, Sobolev and Hölder spaces and their standard properties, as well as with the more general $L^p$ spaces of functions with values in general Banach spaces and with anisotropic Sobolev spaces. Throughout the paper, $(\cdot, \cdot)_{L^p}$ denotes the standard scalar product (norm) in $L^2(\Omega)$, while $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $H^1(\Omega)$ and its dual $(H^1(\Omega))^\ast$. For $p \in [1, \infty] \setminus \{2\}$, we write $\| \cdot \|_{L^p}$ in place of the $\| \cdot \|_{L^p(\Omega)}$-norm. Sometimes, it is useful to consider a local version of an $L^p$ or a Sobolev space. For example, for $p \in [1, \infty]$, a set $I \subset \mathbb{R}^m$ and a Banach space $X$, a local version of a general $L^p$ space is the space $L^p_{loc}(I; X) := \{u : I \to X \mid u \in L^p(K; X)\}$ for all compact $K \subset I$.

Partial derivatives, in both classical and distributional sense, with respect to the variable $t$, we denote by $\partial_t$, while $\nabla$, $\nabla \cdot$ and $\Delta$ stand for the spatial gradient, divergence and Laplace operators, respectively. $\partial_v$ is the derivative with respect to the outward unit normal of $\partial \Omega$.

Finally, we make the following useful convention. For all indices $i$, $C_i$ stands for a non-negative constant or, alternatively, a non-negative non-decreasing function of its arguments.

2.2. Problem setting

In this section, we introduce a PDE system for two variables: the nutrient concentration $n$ and the biomass (contains the bacteria and the surrounding matrix) density $b$, both depending on time and position on a domain $\Omega \subset \mathbb{R}^N$. Our system, a generalisation of a system introduced in [3], reads:

\begin{align}
\partial_t n &= d_n \Delta n - f_n(n, b) \text{ in } (0, T) \times \Omega, \\
\partial_t b &= \nabla \cdot \left( d_b n^\beta b^\alpha \nabla b \right) + f_b(n, b) \text{ in } (0, T) \times \Omega, \\
\partial_v n &= 0, \quad n^\beta \partial_v b^\alpha + 1 = 0 \text{ in } (0, T) \times \partial \Omega, \\
n(0) &= n_0, \quad b(0) = b_0 \text{ in } \Omega.
\end{align}

The nutrients are assumed to diffuse with a constant diffusion coefficient $d_n > 0$. For the movement of the bacteria, we use a nonlinear power law diffusion coefficient

\[ D_b(n, b) := d_b n^\beta b^\alpha \text{ for some } d_b, \alpha, \beta > 0. \]

This choice is motivated by the fact that the more nutrients and bacteria are around, the better is the mobility of a bacterial cell. Recall that, in particular, the porous-medium type degeneracy ($D_b(n, 0) = 0$) ensures a finite speed of propagation for the bacteria, in contrast to the infinite speed of propagation of the nutrient. The positive powers $\alpha$ and $\beta$ depend on the properties of the bacterial species. These powers can be interpreted as weights. They describe whether the bacterial diffusion is more influenced by the local concentration of nutrients or, rather, by the local bacterial density. The positive constants $d_n$ and $d_b$ mainly depend on the density of the agar on the plate. In general, the more dense the agar, the harder becomes the spread-out and, as a result, the smaller are $d_n$ and $d_b$.

The nutrients are consumed by the bacteria, this is described by the non-negative term $f_n(n, b)$. Both $f_n(n, b)$ and the bacterial growth term $f_b(n, b)$ are able to cover a rather general nonlinear dependence upon $n$ and $b$. The usual choice is that the bacterial growth is proportional to the nutrient consumption. But the term $f_b(n, b)$ can also include a switch to the quiescent state, which is neglected here for the moment.

On the boundary, no-flux conditions are assumed, which is realistic, since, in the typical experimental situation, the rims of a Petri dish prevent any flux.
This paper is devoted to the well-posedness study of system (2.1). We make the following basic assumptions on the parameters of the problem and the initial data:

Assumptions 2.1

(1) \( \Omega \) is a bounded domain of class \( C^2 \) in \( \mathbb{R}^N \), \( N \in \mathbb{N} \).

(2) \( d_n, d_b, \alpha, \beta \) are positive constants;

(3) The nonlinearities \( f_n, f_b \) satisfy

\[
\begin{align*}
&f_n, f_b \in W^{1,\infty}(\mathbb{R}_+^N \times \mathbb{R}_+^N), \\
&f_n(0, b) = 0 \text{ for all } b \in \mathbb{R}_+^N, f_b(n, 0) = 0 \text{ for all } n \in \mathbb{R}_+^N;
\end{align*}
\]

(4) The initial data \( n_0, b_0 \) satisfy \( n_0 \in W^{1,\infty}(\Omega) \), \( b_0 \in L^\infty(\Omega) \) and are non-negative.

Under Assumptions 2.1, there exists a local solution to problem (2.1), see Lemma 3.1 below. To ensure global existence, we impose the following additional

Assumptions 2.2

(1) \( f_n \) and \( f_b \) satisfy

\[
\begin{align*}
&- F_n f_n(n, b) + f_b(n, b) \leq F_M (F_n n + b) \text{ for all } n, b \in \mathbb{R}_+^N, \\
&0 \leq f_n(n, b) \leq f_1(n)n^{\alpha + 1}(b^{c_1 + 1} + 1) \text{ for all } n, b \in \mathbb{R}_+^N, \\
&f_b(n, b) \leq f_2(n)(b^{c_2 + 1} + 1) \text{ for all } n, b \in \mathbb{R}_+^N
\end{align*}
\]

for some constants \( a > 0, c_1, c_2, F_n, F_M \geq 0 \) and non-decreasing functions \( f_1, f_2 \in C(\mathbb{R}_+^N, \mathbb{R}_+^N) \);

(2) \( N, \alpha, \beta, a, c_1, c_2 \) together satisfy the 'subcritical' growth assumption

\[
\frac{\beta}{a} \leq \frac{\alpha + \frac{2}{N} - c_2}{c_1 + 1}.
\]

Under the assumptions (2.3) and (2.4), system (2.1) has the following two important properties: the non-negativity of the solutions is preserved for all time (this is a consequence of [5, Theorem 2.3]), and the ‘total masses’ of the components, i.e., the norms \( \|n\|_1, \|b\|_1 \), are a priori bounded on all finite time intervals (the so-called ‘mass’ control, see, e.g., [6]). In general, without any further restrictions imposed on the source terms, there may be no global solutions of the system (2.1). In this paper, we formulate a sufficient condition for global existence in terms of the subcritical growth condition (2.7). We call this condition subcritical in the sense that if it is fulfilled, the solutions of (2.1) exhibit no blowup in finite time: they exist globally and stay bounded on all finite time cylinders. For \( c_1 = c_2 =: c \), the growth condition (2.7) simplifies to:

\[
c + 1 \leq \frac{N + 2}{1 + c}.
\]

This is an analog of the well-known growth assumption

\[
c + 1 < \frac{N + 2}{N},
\]

generally used in the study with bootstrap arguments of semilinear systems with the mass control property (see, e.g., the study of system (1.5) in [7]). In our setting, the semilinear case would correspond to the case \( \alpha = \beta = 0 \) (not included in this study). We note, however, that in this limit situation the inequality in the growth assumption (2.9) is strict.

For uniqueness of solutions, further restrictions are needed:
Assumptions 2.3

1) \( N \in \{1, 2, 3\} \);
2) \( \beta \geq 2 \);
3) \( f_n \) satisfies

\[
 f_n(n, b) = \int_0^b f_{11}(n, \omega) n^{\frac{\alpha}{\beta}} \omega^{\frac{\alpha}{\beta}} \, d\omega + f_{12}(n) \quad \text{for all } n, b \in \mathbb{R}_0^+ \tag{2.10}
\]

for some functions \( f_{11} \in L^\infty_{\text{loc}}(\mathbb{R}_0^+ \times \mathbb{R}_0^+) \), \( f_{12} \in W^{1,\infty}_{\text{loc}}(\mathbb{R}_0^+) \) with \( f_{12}(0) = 0 \);
4) \( f_b \) satisfies

\[
 f_b(n, b) = \int_0^b f_{21}(n, \omega) n^{\frac{\alpha}{\beta}} \omega^{\frac{\alpha}{\beta}} \, d\omega + F_{22} b \quad \text{for all } n, b \in \mathbb{R}_0^+ \tag{2.11}
\]

for some function \( f_{21} \in L^\infty_{\text{loc}}(\mathbb{R}_0^+ \times \mathbb{R}_0^+) \) and constant \( F_{22} \in \mathbb{R} \).
5) \( b^{n+1} \in H^1(\Omega) \).

The shape of the reaction terms \( f_n \) and \( f_b \) differs considerably from the standard growth kinetics terms used in most similar studies. In this study the division rate is dependent on the biomass density \( b \). In the important case \( F_{22} = 0 \), there is, due to \( \alpha > 0 \), a delay in the biomass growth wherever \( b \) is close to 0, especially during the early stages of bacterial growth. This corresponds to a ‘lag-phase’ needed during the onset of the growth of bacterial communities.

We define the weak solutions of (2.1) in the following way:

**Definition 2.4** We call a pair of non-negative functions \((n, b)\) defined in \([0, T_\ast) \times \Omega\) for some \( T_\ast \in (0, \infty) \) a weak solution of (2.1) for some non-negative \( n_0 \in W^{1,\infty}(\Omega) \), \( b_0 \in L^\infty(\Omega) \) if for all \( 0 < \tau < T < T_\ast \)

1) \( n \in L^\infty((0, T); W^{1,\infty}(\Omega)) \), \( b \in L^\infty((0, T); L^\infty(\Omega)) \);
2) \( n \in C^{1,2}([\tau, T] \times \Omega) \), \( n^\beta \nabla b^{n+1} \in L^\infty((\tau, T); L^2(\Omega)) \), \( \partial_t b \in L^\infty((\tau, T); H^1(\Omega)) \);
3) \((n, b)\) satisfies the equation (2.1a) and \( n \) satisfies the condition \( \partial_t n = 0 \) in the classical sense;
4) \((n, b)\) satisfies the equation (2.1b) and the condition \( n^\beta \partial_t b^{n+1} = 0 \) in the following sense: for all \( \varphi \in H^1(\Omega) \) it holds

\[
 \langle \partial_t b, \varphi \rangle = -d_b \left( n^\beta \nabla b^{n+1}, \nabla \varphi \right) + (f_b(n, b), \varphi) \quad \text{a.e. in } (0, T);
\]
5) \( n(0) = n_0, b(0) = b_0 \).

We further say that a solution \((n, b)\) is regular if, in addition to the conditions (i)-(v), it holds for all \( 0 < T < T_\ast \) that

\[
 b^{n+1} \in L^\infty([0, T], H^1(\Omega)).
\]

A solution is called global if it exists in the whole of \([0, \infty) \times \Omega\) and local if otherwise.

**Remark (Initial conditions)** Since we are looking for solutions \((n, b)\) with \( n \in L^\infty((0, T); W^{1,\infty}(\Omega)) \), \( b \in L^\infty((0, T); L^\infty(\Omega)) \), it follows from (2.1a)-(2.1c) that \( \partial_t n \in L^\infty((0, T); W^{1,1}(\Omega)) \), \( \partial_t b \in L^\infty((0, T); W^{2,1}(\Omega)) \). Therefore, due to [8, Chapter II, §1, Theorem 1.5] and [9, Chapter 3, Lemma 1.4], we have \( n \in C([0, T] \times \Omega) \), \( b \in C_w([0, T]; L^\infty(\Omega)) \), so that the initial conditions (v) do make sense. (Recall that \( C_w([0, T]; L^\infty(\Omega)) \) denotes the space of functions \( u : [0, T] \rightarrow L^\infty(\Omega) \) which are continuous with respect to the weak topology of \( L^\infty(\Omega) \)).

Our main result reads:

**Theorem 2.6 (Existence and uniqueness)** Let \( \Omega \subset \mathbb{R}^N \), \( N \in \mathbb{N} \), be a bounded domain of class \( C^2 \) and let \( d_n, d_b, \alpha, \beta > 0 \).
Consider the system

\[ B \text{Assumptions } 2.1-2.3 \]

Therefore, if \( t \geq 0 \), \( b_0 \in L^\infty(\Omega) \), and \( (2.11) \) is fulfilled, then there exists a regular global weak solution.

(2) **(Uniqueness)** Assume that \( N \in \{1, 2, 3\} \), \( \beta \geq 2 \), and the functions \( f_n, f_b \) satisfy \((2.10)-(2.11)\). Then, for all non-negative initial data \( n_0 \in W^{1,\infty}(\Omega) \), \( b_0 \in L^\infty(\Omega) \) with \( b_0 \in H^1(\Omega) \) and \( \beta \geq 1 \), there exists at most one regular weak solution of \((2.1)\).

Therefore, if Assumptions 2.1-2.3 are fulfilled, system \((2.1)\) is globally well-posed.

Let us now consider a parameter dependent example.

**Example 2.7** Consider the system

\[
\begin{align*}
\partial_t n &= d_n \Delta n - \frac{G_n b^{1+n+a_1}}{1 + \gamma n^{1+a_2}} \text{ in } (0, T) \times \Omega, \\
\partial_t b &= \nabla \cdot \left( d_n b a^n \nabla b \right) + G_b b^{1+n+a_1} \text{ in } (0, T) \times \Omega, \\
\partial_t \nu n &= 0, \quad \nu \beta b^{n+1} = 0 \text{ in } (0, T) \times \partial \Omega, \\
(n(0) = n_0, \quad b(0) = b_0) \text{ in } \Omega
\end{align*}
\]  

for some constants \( \alpha, \beta, G_1, G_2 > 0, a_1, a_2, c > 0 \). In this example, the bacterial growth \( f_n(n, b) \) is proportional to the nutrient consumption of the form

\[ f_n(n, b) = G_n \frac{b^{1+n+a_1}}{1 + \gamma n^{1+a_2}} \]

with the conversion rate \( \frac{G_n}{\gamma} \).

Of special interest is the two dimensional case, i.e., \( N = 2 \). Then the growth condition \((2.8)\) simplifies to \((c + 1)(1 + \frac{\alpha}{2}) \leq 2 + \alpha \). In Section 5, we discuss two cases numerically and illustrate the influence of the parameter choice on pattern formation.

1. \( a_2 := \frac{\alpha}{2}, \quad c := \frac{\alpha}{2}; \quad \alpha, \beta, a_1 := a \) vary. The growth condition \((2.8)\) together with Assumptions 2.3 are equivalent to

\[ a \geq \beta \geq 2. \tag{2.13} \]

Here we have three parameters, where \( \alpha \) and \( \beta \) enter into the exponent and control the dynamic of the bacteria. The higher \( \alpha \), the more we relate the bacterial movement to the bacterial density. The higher \( \beta \), the more it is related to the nutrient availability. With the parameter \( a \) we are able to influence the growth term of the bacteria and again give the nutrients a higher importance for bacterial growth.

2. \( \alpha := \beta := 1, \quad a_2 = a_1 := a; \quad a, c \) vary. The condition \((2.8)\) is equivalent to

\[ a \geq \frac{1 + c}{2 - c}. \tag{2.14} \]

This case is a generalization of the model in [3], where the parameters \( \alpha = \beta = 1, \quad a_1 = a_2 = c = 0 \) can be found. By choosing \( \alpha = \beta = 1 \), we give equal weights to the influence of the nutrient concentration and the bacterial density on the bacterial spreading. In contrast to [3], having \( a \) and \( c \) as independent parameters allows us to influence the growth term. Depending on the type of bacteria and the resulting influence of the bacterial density or the nutrients on the bacterial growth, it is appropriate to change either the term \( c \) or \( a \). However, we have to note that the choice \( a_1 = a_2 = c = 0 \) and \( \beta = 1 \) does not satisfy Assumption 2.3(2) and \((2.8)\). Therefore we cannot apply our Theorem 2.6 to this case. Still, due to \( c = 0 \), the growth is at most linear in \( b \), so that, in this special case, no blow-up occurs in finite time for any non-negative diffusion coefficient. We observe numerically in Section 5 that a fingering structure occurs typically if and only if the growth condition is violated to some extent, see also Remark 5.2.
2.3. Comparison with a system with a porous medium-type degeneracy

To gain a better understanding of the meaning of the conditions imposed on the reaction terms \( f_n \) and \( f_b \), let us compare the system (2.1) with a system governed by the classical porous medium degeneracy:

\[

c_t n = d_n \Delta n - f_n(n, b) \quad \text{in} \ (0, T) \times \Omega, \\
\dot{c}_b = \nabla \cdot (d_b n \nabla b) + f_b(n, b) \quad \text{in} \ (0, T) \times \Omega, \\
\dot{c}_n n = 0, \quad \dot{c}_n b^{n+1} = 0 \quad \text{in} \ (0, T) \times \partial \Omega, \\
\n(0) = n_0, \quad b(0) = b_0 \quad \text{in} \ \Omega.
\]

(2.15a)

(2.15b)

(2.15c)

(2.15d)

Clearly, the difference between these two systems lies in the equations (2.1b) and (2.15b) for the \( b \) component. In (2.1b), the variable \( n \) appears only in the source term \( f_b(n, b) \). It is, therefore, a case of the so-called weak coupling. In (2.1b), however, this variable appears in the diffusion coefficient as well, so that, the evolution equations for \( n \) and \( b \) are strongly coupled. For both systems, the existence of weak local solutions is readily established. Let \([0, T_{\text{max}}], T_{\text{max}} \in (0, \infty), \) be the maximal existence interval of such solution. It is well understood that \( T_{\text{max}} < \infty \) if and only if \( \lim_{t \to T_{\text{max}}} \| b(t) \|_{\infty} = \infty \), and, in such case, the solution fails to exist globally. Thanks to the zero Neumann boundary condition, the \( n \) component for \( n_0 \neq 0 \) is actually bounded from below by a positive constant uniformly on \( \Omega \) on each time interval \([\tau, t]\), \( 0 < \tau < t < T_{\text{max}} \). However, due to the decrease of \( \| n \|_{\infty} \) in the region where \( f_n(n, b) > 0 \), the diffusion coefficient in (2.1b) may tend to zero as \( t \to T_{\text{max}} < \infty \) even for \( b \to \infty \). This, is the case when \( n^2(t_m, x_m) = o(b^{-\gamma}(t_m, x_m)) \) for some sequence \( \{t_m, x_m\} \) \( \in [0, T_{\text{max}}] \times \Omega \) with \( t_m \to T_{\text{max}} \) as \( m \to \infty \). Consequently, even in the ‘subcritical’ ‘superlinear’ case, when the growth order in \( b \) in the term \( f_b \) is more than linear but less than \( b^{1+\alpha+\beta} \) as \( b \to \infty \), so that each solution of (2.15) exists globally (the proof is similar to the one given in [7] for a semilinear system), solutions of (2.1) may exhibit blowups in finite time. The ‘subcritical condition’ (2.7) that we proposed above, can be seen as substitute of the subcritical growth assumption in the classical case when \( \beta = 0 \). It proves to be sufficient for the existence of bounded solutions for \( \beta > 0 \).

The uniqueness of solutions for the system (2.15) is a consequence of the monotonicity of the principle parts of the system in \( L^1 \). In our case, the dependence of the diffusion coefficient in the equation for \( b \) upon another variable (\( n \) in this case) forces us to work in spaces other than \( L^1(\Omega) \). This results in considering the nonlinearities \( f_n \) and \( f_b \) with ‘right’ behaviour near the degeneracy level \( b = 0 \). Moreover, since the initial value \( n_0 \) is not necessarily sharply separated from 0, and since the equations (2.1a) and (2.1b) are strongly coupled, the analysis becomes more involved. It requires from the nonlinearities the ‘right’ behaviour of \( f_n \) and \( f_b \) near the level \( n = 0 \).

The following remark gives a brief comparison of (2.1) and (2.15) with regard to positivity propagation and regularity of solutions.

The careful analysis of these properties is beyond the scope of this paper.

Remark 2.8 (Positivity and regularity)

1. It was observed above that, for both systems (2.1) and (2.15), the function \( n \) is positive in the whole cylinder \((0, T_{\text{max}}) \times \bar{\Omega}\) (if \( n_0 \neq 0 \)). Thus, its speed of propagation of positivity is infinite in both cases. This is a classical property of semilinear parabolic equations. It is well-known [10] that the speed of propagation of positivity of solutions of a degenerate parabolic equation like (2.1b) or (2.15b) is finite, which means that the trace of a solution doesn’t instantly (i.e., as soon as \( t > 0 \)) coincide with \( \bar{\Omega} \). Since the function \( n \) is a priori bounded from above by \( \| n_0 \|_{\infty} \) due to the comparison principle, the positivity expansion is, roughly speaking, ‘no faster’, than it would be for a diffusion coefficient \( d_n \| n_0 \|_{\infty}^{\frac{\beta}{\beta+\gamma}} \) in place of \( d_n n^{\beta+\gamma} \). Moreover, it can even be ‘slower’ in the regions where \( n \) is small. Therefore, the local interaction between functions \( n \) and \( b \) which solve (2.1) does not only influence the boundedness, but also the propagation speed of the biomass. The numerical simulations in Section 5 convey some impression of the form that the trace of the biomass takes.

2. The function \( n \) which solves a semilinear parabolic equation like (2.1a) or (2.15a), instantly becomes smooth and is actually a strong solution. This is not the case for a function \( b \) solving (2.1b) or (2.15b), since, as is well-known [10], the solutions of a degenerate equation are, in general, only Hölder continuous globally. We recall that the Hölder exponent and constant for a degenerate equation are difficult to estimate. In particular, in the case of equation (2.1b), their local estimates would involve not only the degeneracy exponent \( \alpha \) and some bound for the reaction term, but also a lower bound for \( n \). Thus, in
the regions where \( n \) is small the solution could be even 'rouger', than it would be for the diffusion coefficient independent from \( n \).

3. A priori estimates and existence of solutions

In this section, we prove the existence and boundedness of global solutions to the system (2.1). We start with the preliminary Lemma 3.1. In this lemma, we show for general non-negative initial data \( (n_0, b_0) \) the existence of a bounded local solution \( (n, b) \) on some finite cylinder \([0, T_1] \times \Omega\). Since we are dealing with the homogeneous Neumann boundary conditions, the \( n \) component of the local solution is uniformly bounded from below in \( \Omega \) by a positive constant for all \( \tau \in (0, T_1] \). Therefore, in the global existence proof, we concentrate on the existence of global solutions starting from initial data with a strictly positive \( n \)-component. An extension of the local solution \( (n, b) \) by a global solution starting at \((n(T_1), b(T_1))\) yields a solution for the initial data \( (n_0, b_0) \) on \([0, \infty) \times \Omega\).

Given the diagonal structure of the degenerate system (2.1), both local and global solutions can be obtained as limits of solutions of standard non-degenerate approximations, provided that suitable uniform a priori bounds hold. The key difficulty lies in establishing the global a priori estimates. This is the main part of the proof. In the proofs below, the statement that a constant depends upon the parameters of the problem means that it depends on the constants \( d_n, d_0, \alpha \) and \( \beta \), the structure of the functions \( f_n \) and \( f_g \), the space dimension \( N \) and the domain \( \Omega \). This dependence upon the parameters is not indicated explicitly.

We thus begin our study of system (2.1) with a local existence result under rather moderate assumptions (2.2)-(2.3) on the source terms.

**Lemma 3.1 (Local existence)** Let \( \alpha, \beta, d_n, d_b > 0 \) and assume that the functions \( f_n, f_b \) satisfy the assumptions (2.2)-(2.3). Then, for all \( n_0 \in W^{1,\infty}(\Omega) \), \( b_0 \in L^{\infty}(\Omega) \), \( n_0, b_0 \geq 0 \) there exists a \( T_1 > 0 \) and a bounded solution \((n, b)\) of system (2.1) in \([0, T_1] \times \Omega\) in terms of Definition 2.4.

**Proof.** Let us first treat the special case \( n_0 \equiv 0 \). Consider the ODE

\[
\begin{aligned}
\dot{c}_b &= f_b(0, b) \quad \text{in } (0, T) \times \Omega, \\
br(0) &= b_0 \quad \text{in } \Omega.
\end{aligned}
\]  
(3.1)

Due to the assumptions (2.2)-(2.3) and the Peano existence theorem, this ODE possesses a non-negative bounded local solution \( b \) defined in a cylinder \([0, T_1] \times \Omega\) for some \( T_1 > 0 \). Then, obviously, the pair \((0, b)\) is a local solution of (2.1).

The proof of the general case \( n_0 \neq 0 \) follows a standard schema. It consists of the following three steps: construction of suitable non-degenerate approximations, checking uniform local a priori estimates for the solutions and, finally, applying the compactness method [11].

We first construct a suitable family of regularised problems: for all \( T > 0 \) and \( m \in \mathbb{N} \) let

\[
\begin{aligned}
\dot{c}_n &= d_n \Delta n_m - f_n(n_m, b_m) \quad \text{in } (0, T) \times \Omega, \\
\dot{c}_b &= \nabla \cdot \left( d_b D^{(m)}_b \left(n_m, b_m^m, \nabla b_m \right) \right) + f_b(n_m, b_m) \quad \text{in } (0, T) \times \Omega, \\
\dot{c}_n &= 0, \quad D^{(m)}_b \left(n_m^m, b_m^m \right) \partial_n b_m = 0 \quad \text{in } (0, T) \times \partial \Omega, \\
n_m(0) &= n_0, \quad b_m(0) = b_0 \quad \text{in } \Omega.
\end{aligned}
\]  
(3.2a, 3.2b, 3.2c, 3.2d)

where

\[
D^{(m)}_b : \mathbb{R}_+^n \to [m^{-1}, m + m^{-1}], \quad D^{(m)}_b(x) := m^{-1} + m - (m - x)_+.
\]
For each such diagonal non-degenerate quasilinear system, the general result from [12] and the positivity result [5, Theorem 2.3] provide the existence of a smooth uniformly bounded local solution

\[
(n_m, b_m) : [0, T_1] \to [0, R_n] \times [0, R_b],
\]

\[
T_1 := \frac{1}{2} \left| \left( f_n, f_b \right) \right|_{L^\infty((0,R_n) \times (0,R_b))}, \quad R_n := \|n_0\|_\infty + 1, \quad R_b := \|b_0\|_\infty + 1.
\]

As a consequence of the classical regularity theory for linear parabolic equations [12], this uniformly bounded solution sequence possesses a subsequence which converges in \( L^\infty((0, T_1) \times \Omega) \) strongly in the \( n \)-component. To deduce the existence of a subsequence with strongly converging \( b \)-component, we have to check the existence of a uniform lower bound for \( n_m \) on each subcylinder \([\tau, T_1] \times \Omega, \tau \in (0, T_1)\). This would mean that the equations (3.2b) are, with respect to the \( b \)-component, locally uniformly of (generalised) porous medium type. We set

\[
M_1 := \|\hat{c}_n f_n\|_{L^\infty((0,R_n) \times (0,R_b))}.
\]

Using the comparison principle for parabolic equations (see, e.g., [13, Chapter 2, Theorem 2.9]), \( n_m \) can be estimated from below by the solution of the linear problem

\[
\begin{aligned}
\hat{c}_t u &= d_n \Delta u - M_1 u \text{ in } (0, T_1) \times \Omega, \\
\hat{c}_t u &= 0 \text{ in } (0, T_1) \times \partial \Omega, \\
u(0) &= n_0 \text{ in } \Omega.
\end{aligned}
\]

The solution reads:

\[
u(t, x) = e^{-M_1 t} \int_\Omega K(t, x, y) u_0(y) \, dy \text{ in } (0, T_1) \times \Omega,
\]

where \( K \) is the heat kernel for homogeneous Neumann boundary conditions. Due to the parabolic Hopf Lemma (see, e.g., [14, Chapter 3, Theorem 8]), \( K \) is bounded from below on \([\tau, T_1] \times \Omega \times \Omega\) by some positive non-decreasing function \( k_1 = k_1(\tau)\). Together with (3.4), it provides a uniform lower bound for the solutions:

\[
n_m(t, x) \geq u(t, x) \geq e^{-M_1 T_1} k_1(\tau) \|n_0\|_1 \text{ for all } t \in [\tau, T_1] \times \Omega.
\]

As a consequence of (3.5), we have for the diffusion coefficient in (3.2b) the following estimate:

\[
D_b^{(m)} \left( n_m^b b_m^a \right) \geq (\alpha + 1) \left( e^{-M_1 T_1} k_1(\tau) \|n_0\|_1 \right) \|b_m^a\|^\beta_{b_m^a} \text{ for all } m \in \mathbb{N}, \quad m \geq R_1^\alpha R_0^\beta.
\]

With estimates (3.3) and (3.6), the classical regularity results for linear parabolic equations [12] and the (generalised) porous medium equation [10], we then conclude that

\[
\|n_m\|_{L^\infty((0, T_1) \times \Omega)} \leq M_2
\]

and

\[
\|n_m\|_{C^\left(1+\frac{\gamma(\tau)}{T_1}, 2+\gamma(\tau)\right)([\tau, T_1] \times \Omega)}, \quad \|b_m\|_{C^\left(1+\frac{\gamma(\tau)}{T_1}, 2+\gamma(\tau)\right)([\tau, T_1] \times \Omega)}, \quad \|b_m^{\alpha+1}\|_{L^\infty([\tau, T_1] \times \Omega)} \leq k_2(\tau) \text{ for all } \tau \in (0, T_1)
\]

for some positive constant \( M_2 \) and non-decreasing functions \( \gamma = \gamma(\tau) \in (0, 1) \) and \( k_2 = k_2(\tau) > 0 \) that depend on \( T_1 \), the initial data and the parameters of the problem, but do not depend on \( m \). These uniform estimates allow us to use standard compactness arguments in order to pass to the limit in the regularising (sub)sequence, and justify that the limit is indeed the solution of the original problem in terms of Definition 2.4. We omit these technical details.
Now we are ready to prove the existence part of Theorem 2.6. As in the case of the lemma above, the main issue here is to obtain an a priori upper bound for the $b$-component and a lower bound for the $n$-component. However, this time the bounds need to be global. The boundedness proof is based on the bootstrap argument: starting from uniform (in time) $L^1(\Omega)$ bounds, we bootstrap to obtain a uniform bound in $L^\infty(\Omega)$. The most important assumptions that we use here are (2.4) and (2.7).

**Proof of Theorem 2.6 (Global existence).** We first treat the special case $n_0 \equiv 0$. As in the proof of Lemma 3.1, we consider the auxiliary ODE (3.1). As a consequence of assumptions (2.4) and $f_n(0, b) = 0$, we have $f_n(0, b) \leq F_M b$ for all $b \in \mathbb{R}^+_0$.

Combined with the assumptions (2.2)-(2.3), the latter leads to the a priori boundedness and non-negativity of solutions for (3.1) on all finite cylinders. Hence, due to the Picard-Lindelöf theorem, (3.1) possesses a (unique) global solution $b$. Then, obviously, the pair $(0, b)$ is a global solution of (2.1).

We now move to the general case $n_0 \not\equiv 0$. From Lemma 3.1, we already know that there exists a $T_1 > 0$ and a corresponding solution $(n, b)$ defined in $[0, T_1] \times \Omega$. Our goal is now to extend the local solution $(n, b)$ defined in $[0, T_1] \times \Omega$ to a global solution on the whole of $[0, \infty) \times \Omega$. Since we are dealing with an autonomous system, this can be done by showing the existence of a global solution of (2.1) with the new initial data: $(n(T_1), b(T_1))$. As in the existence proof from the above lemma, our main task here would be to obtain an upper bound for the $b$-component and a lower bound for the $n$-component. In the proof of Lemma 3.1, we actually showed (see (3.5)) that if $n_0 \not\equiv 0$, then, for each $\tau \in (0, T_1]$, the function $n(\tau)$ is continuous and strictly positive on $\bar{\Omega}$. It then follows directly that

$$n^{-1}(T_1) \in C(\bar{\Omega}). \quad (3.7)$$

This property of $n(T_1)$ plays an important role in what follows.

To begin with, both $n$ and $b$ are a priori bounded in $L^1(\Omega)$ on all finite time intervals $[0, T]$ (the so-called ‘mass control’ [6]). Indeed, due to $f_n \leq 0$, $||n||_1$ is decreasing and, with the assumption (2.4), it follows for the solutions of the system (2.1) that

$$\frac{d}{dt}(F_n||n||_1 + ||b||_1) \leq F_M(F_n||n||_1 + ||b||_1).$$

Using Gronwall’s lemma, we get an estimate for $||b||_1$:

$$||b(\tau)||_1 \leq F_n||n(\tau)||_1 + ||b(\tau)||_1 \leq e^{TF_M} (F_n||n_0||_1 + ||b_0||_1).$$

Moreover, $n$ is even a priori bounded by $||n_0||_\infty$ in $L^\infty(\mathbb{R}^+_0 \times \Omega)$ due to the comparison principle (see, e.g., [13, Chapter 2, Theorem 2.9]) since $f_n \geq 0$ (see (2.5)). Based on the $L^1(\Omega)$-bound for $b$ and the uniform boundedness of $n$, we establish a priori $L^p$-bounds for arbitrary $p > 1$ for both $b$ and $n^{-1}$. This idea of propagation of $L^p$ bounds goes back to Alikakos [7].

Let $p_1 \geq a$ and $p_2 > 1$ be arbitrary but fixed. We begin our analysis with the equation (2.1a) and multiply it by $-p_1 n^{-p_1 - 1}$.

Formal integration by parts yields

$$\frac{d}{dt} ||n^{-1}||_{p_1} \leq -\frac{4d_{n}(p_1 + 1)}{p_1} \left|\nabla n^{-\frac{p_1}{2}}\right|^2 + p_1 \left(n^{-p_1-1}, F_n(n, b)\right).$$

Using the assumption (2.5) and Hölder’s inequality, we obtain that

$$\frac{d}{dt} ||n^{-1}||_{p_1} \leq p_1 f_1(||n||_\infty) \left(n^{-p_1-1}, b^{a+1} + 1\right) \leq p_1 f_1(||n||_\infty) \left(n^{-p_1-1}, b^{a+1} + 1\right).$$

(3.8)
Next, we turn to equation (2.1b). We multiply it by $p_2 b^{p_2 - 1}$ and integrate by parts to obtain that

$$
\frac{d}{dt} |b^{p_2}|_1 = -\frac{4d_p p_2 (p_2 - 1)}{(p_2 + \alpha)^2} \left| n^{\frac{\alpha}{2}} \nabla b \frac{p_2 + \alpha}{p_2 + \alpha} \right|^2 + p_2 \left| (b^{p_2 - 1}, f_t(n, b)) \right| .
$$

(3.9)

We then infer from (3.9), (2.6) and Hölder’s inequality that it holds:

$$
\frac{d}{dt} |b^{p_2}|_1 \leq - C_1 \left| n^{\frac{\alpha}{2}} \nabla b \frac{p_2 + \alpha}{p_2 + \alpha} \right|^2 + p_2 f_2(|n||_\infty) \left( (b^{p_2 - 1}, b^{p_2 + 1}) \right)
$$

$$
\leq - C_1 \left| n^{\frac{1}{p_1}} \frac{\beta}{p_1} \frac{p_2}{p_2 + \alpha} \right|^2 C_2(p_2, |n||_\infty) \left( \left| b \right|^\frac{p_2 + \alpha}{p_2 + \alpha} \right)^2 + 1 ,
$$

(3.10)

In order to estimate the norms of $b^{p_2 + \alpha}$ on the right side of (3.8) and (3.10), we need the interpolation inequality

$$
|u^p|_{q, r} \leq C_3(|\nabla u^p|, |u|^{p_1}|u||^{1-\theta}p) \text{ for } u^p \in W^{1, r} \Omega, \ \theta := \frac{p - q}{p - q + \frac{1}{r}}
$$

for $p, q, r \geq 1$ such that $\theta \in [0, 1)$. This is a simple consequence of the standard interpolation inequality for Sobolev spaces [15, Chapter IV, Theorem 4.17]

$$
|v|_q \leq |v|_{W^{1, r}(\Omega)} |v|^{1-\theta} \text{ for } v \in W^{1, r} \Omega, \ \theta := \frac{1 - \frac{1}{p}}{1 - \frac{1}{p} + \frac{1}{r}}
$$

for $v := u^p$ combined with the interpolation inequality

$$
|u|_p \leq |u|_{p_2^*} |u|^{1-\theta_2}, \ \theta_2 := \frac{1 - \frac{1}{p}}{1 - \frac{1}{p_2}}
$$

and the Poincaré inequality. With the help of (3.11), we obtain that

$$
\left| b^{p_2 + \alpha} \right|_{2, 2} \frac{c_1 + 1}{p_1 + \alpha} \leq C_4(p_2) \left( \left| b \right|^{1-\theta_1} |c_1 + 1| \left| \nabla b \frac{p_2 + \alpha}{p_2 + \alpha} \right|_{2, 2} + \left| b \right|_{1, 1} \right), \ \theta_1 := \frac{p_2 + \alpha - \frac{c_1 + 1}{p_2 + \alpha}}{p_2 + \alpha - 1 - \frac{1}{p_1} + \frac{1}{r}},
$$

(3.12)

$$
\left| b^{p_2 + \alpha} \right|_{2, 2} \frac{c_2 + 1}{p_2 + \alpha} \leq C_5(p_2) \left( \left| b \right|^{1-\theta_2} |c_2 + 1| \left| \nabla b \frac{p_2 + \alpha}{p_2 + \alpha} \right|_{2, 2} + \left| b \right|_{1, 1} \right), \ \theta_2 := \frac{p_2 + \alpha - \frac{c_2 + 1}{p_2 + \alpha}}{p_2 + \alpha - 1 - \frac{1}{p_1} + \frac{1}{r}},
$$

(3.13)

provided that it holds (due to the assumptions on the parameters for (3.11) from above):

$$
p_2 + \alpha \geq 2,
$$

(3.14)

$$
\frac{p_1}{p_1 + \beta} \leq \alpha \leq \frac{p_2 + \alpha}{p_2 + \alpha} \geq \frac{1}{2},
$$

(3.15)

$$
\theta_1, \theta_2 \in [0, 1).
$$

(3.16)

Let us assume, for a moment, that $p_1$ and $p_2$ can be chosen in such a way that (3.14)-(3.16) holds. Then, adding (3.8) and (3.10), using (3.12)-(3.13) and Young’s inequality, we get

$$
\frac{d}{dt} \left( \left| b^{p_2} \right|_1 + \left| n^{-1} \right|_{p_1} \right)
$$

$$
\leq - C_1 \left| n^{-1} \right|_{p_1} \frac{\beta}{p_1} \frac{p_2}{p_2 + \alpha} + C_6(p_1, p_2, \left| n \right|_{\infty})
$$

$$
\left( \left| n^{-1} \right|_{p_1} \left( \left| b \right|^{1-\theta_1} |c_1 + 1| \left| \nabla b \frac{p_2 + \alpha}{p_2 + \alpha} \right|_{2, 2} + \left| b \right|_{1, 1} \right) + \left| b \right|^{1-\theta_2} |c_2 + 1| \left| \nabla b \frac{p_2 + \alpha}{p_2 + \alpha} \right|_{2, 2} + \left| b \right|_{1, 1} \right),
$$

(3.17)
\[ \leq -C_7 \left| n^{-1} \right|^{-\beta} \left\| \nabla b^{\frac{p_2+\alpha}{p_1}} \right\|_{p_1}^2 + C_9(p_1, p_2, \left| n \right|_{\infty} \left\| b \right\|_1) \left( 1 - \theta_1 \frac{c_1+1}{p_2+\alpha} \left| n^{-1} \right|_{p_1}^{-\beta} \left\| b \right\|_1 \right) + C_5(p_1, p_2, \left| n \right|_{\infty} \left\| b \right\|_1) \left( 1 - \theta_2 \frac{c_1+1}{p_2+\alpha} \left| n^{-1} \right|_{p_1}^{-\beta} \left| n \right|_{\infty} \left\| b \right\|_1 \right) + C_6(p_1, p_2, \left| n \right|_{\infty} \left\| b \right\|_1) \left( \left| n^{-1} \right|_{p_1}^{1+1} + \left\| b \right\|_{p_2+c_2} \right) \] (3.17)

whenever

\[ 0 \leq \theta_1 \frac{c_1+1}{p_2+\alpha}, \theta_2 \frac{p_2+c_2}{p_2+\alpha} < 1. \] (3.18)

Moreover, if it holds that

\[ 0 \leq \frac{\beta \theta_1 \frac{c_1+1}{p_2+\alpha} + p_1 - a}{1 - \theta_1 \frac{c_1+1}{p_2+\alpha}}, \frac{\beta \theta_2 \frac{p_2+c_2}{p_2+\alpha}}{1 - \theta_2 \frac{p_2+c_2}{p_2+\alpha}} \leq p_1, \] (3.19)

then it follows from (3.17) that

\[ \frac{d}{dt} \left( \left\| b^{p_2} \right\|_1 + \left| n^{-1} \right|_{p_1}^{p_1} \right) \leq -C_{10} \left| n^{-1} \right|^{-\beta} \left\| \nabla b^{\frac{p_2+\alpha}{p_1}} \right\|_{p_1}^2 + C_{11}(p_1, p_2, \left| n \right|_{\infty} \left\| b \right\|_1) \left( \left| n^{-1} \right|_{p_1} + 1 \right). \] (3.20)

We remark that (3.19) is equivalent to

\[ 0 < \theta_1 \frac{c_1+1}{p_2+\alpha} \leq \frac{a}{p_1+\beta}, 0 < \theta_2 \frac{p_2+c_2}{p_2+\alpha} \leq \frac{p_1}{p_1+\beta}. \] (3.21)

Thus, applying Gronwall’s Lemma to the differential inequality (3.20), we conclude that for those \( p_1 \) and \( p_2 \) that satisfy (3.14)-(3.16) and (3.18),(3.21), it holds for all \( t > T_1 \):

\[ \left\| b^{p_2}(t) \right\|_1 + \left| n^{-1}(t) \right|_{p_1}^{p_1} + \int_{T_1}^{t} \left| n^{-1} \right|^{-\beta} \left\| \nabla b^{\frac{p_2+\alpha}{p_1}} \right\|_{p_1}^2 \, dt \leq C_{11}(p_1, p_2, \left| n \right|_{\infty} \left\| b \right\|_1) \left( \left\| b^{p_2}(T_1) \right\|_1 + \left| n^{-1}(T_1) \right|_{p_1} + 1 \right) < \infty. \] (3.22)

In order to justify (3.22), it thus remains to check that conditions (3.14)-(3.16) and (3.18), (3.21) can be fulfilled. Indeed, they may be reduced to the following set:

\[ p_1 \geq a, \beta, p_2 > 1, 2 - \alpha. \] (3.23)

\[ 1 + \frac{\beta}{p_1} \leq \frac{a}{p_1} \frac{p_2+\alpha}{p_1+1}, \frac{p_2+\alpha}{p_2+c_2} \leq 2. \] (3.24)

\[ \frac{c_1+1}{a} \leq \frac{p_2+\alpha}{p_1+\beta} \leq \frac{\alpha+\frac{2}{N} - c_2}{\beta}. \] (3.25)

Now, it is clear from assumption (2.7) that conditions (3.23)-(3.25) are fulfilled if, for example, we choose

\[ p_2 = \frac{c_1+1}{a} \left( p_1 + \beta \right) - \left( \alpha + \frac{2}{N} \right), p_1 \] sufficiently large.

(3.26)

Hence, choosing \( p_1 \) and \( p_2 \) in accordance to (3.26), we obtain from (3.22) that

\[ b \in L_{\text{loc}}^{\infty}([T_1, \infty); L^p(\Omega)) \text{ for all } p \in [1, \infty), \] (3.27)

\[ n^{-1} \in L_{\text{loc}}^{\infty}([T_1, \infty); L^p(\Omega)) \text{ for all } p \in [1, \infty). \] (3.28)

Combined with (2.5), (3.27) yields

\[ f_n(n, b) \in L_{\text{loc}}^{\infty}([T_1, \infty); L^p(\Omega)) \text{ for all } p \in [1, \infty), \]
which, together with a classical regularity result for parabolic equations [12], gives

$$n \in L^\infty_{\text{loc}}([T_1, \infty); W^{1,\infty}(\Omega)).$$

(3.29)

To obtain an $L^p$-estimate for $n^{-1}$, we observe that due to (2.1a) it satisfies in $(T_1, \infty) \times \Omega$ the equation

$$\partial_t n^{-1} = d_n \Delta n^{-1} + h, \quad h := n^{-2} f_n(n, b) + n^{-3} |\nabla n|^2.$$

(3.28)-(3.29) imply that $h \in L^p_{\text{loc}}((T_1, \infty); L^p(\Omega))$ for all $p \in [1, \infty)$, so that, once again the regularity result for parabolic equations yields

$$n^{-1} \in L^\infty_{\text{loc}}([T_1, \infty); L^\infty(\Omega)).$$

(3.30)

Since we are dealing with a subcritical growth in (2.1b) ($\alpha_2 < \alpha + \frac{2}{\beta}$ due to (2.7)), (3.30) leads to

$$b \in L^\infty_{\text{loc}}([T_1, \infty) \times \Omega).$$

The uniform boundedness of $b$ on each cylinder $(T_1, T) \times \Omega$, $T > T_1$ is thus established. The rest of the existence proof can be done in the same manner as in the proof of Lemma 3.1. We omit these technical details.

It remains to check the existence of a regular solution under the additional assumptions $v_0^{\beta+1} \in H^1(\Omega)$, $\beta \geq 2$ and the assumption (2.11) on the non-linearity $f_b$. Again, it suffices to derive an appropriate a priori estimate. For this purpose, we multiply equation (2.1b) by

$$\frac{\alpha + 2}{2(\alpha + 1)} e^{-(\alpha+1)F_{2t}t} n^{-\beta} \partial_t \big(e^{-F_{2t}t} b\big)^{\alpha+1}$$

and integrate by parts over $\Omega$ to obtain that

$$\frac{2}{\alpha + 2} \left| e^{-\frac{\alpha}{2} F_{2t}t} n^{-\frac{\alpha}{2}} \partial_t \left(e^{-F_{2t}t} b\right)^{\alpha+1} \right|^2 = - \frac{d_n (\alpha + 2)}{2(\alpha + 1)^2} \frac{d}{dt} \left| \nabla \left(e^{-F_{2t}t} b\right)^{\alpha+1} \right|^2 + \frac{2d_n}{\alpha + 1} \left( e^{-\frac{\alpha}{2} F_{2t}t} n^{-\frac{\alpha}{2}} \partial_t \left(e^{-F_{2t}t} b\right)^{\alpha+1}, b \frac{\alpha}{2} n^{-\frac{\alpha}{2}} \nabla \cdot \nabla \left(e^{-F_{2t}t} b\right)^{\alpha+1} \right) + \left( e^{-\frac{\alpha}{2} F_{2t}t} n^{-\frac{\alpha}{2}} \partial_t \left(e^{-F_{2t}t} b\right)^{\alpha+1}, e^{-(\alpha+1)F_{2t}t} b^{-\frac{\alpha}{2}} \left(f_b(n, b) - F_{2t}b\right) n^{-\frac{\beta}{2}} \right).$$

(3.31)

With the help of assumption (2.11), we get the upper bound

$$|f_b(n, b) - F_{2t}b| n^{-\frac{\beta}{2}} \leq C_1 (|n|_{L^\infty(\Omega)}, |b|_{L^\infty(\Omega)}).$$

(3.32)

Using estimate (3.32), the assumptions $\alpha > 0$, $\beta \geq 2$ and the boundedness of global solutions ($n \in L^\infty((0, T); W^{1,\infty}(\Omega))$, $b \in L^\infty((0, T) \times \Omega)$) for all $T > 0$, we conclude from (3.31) with the Cauchy-Schwarz and Young’s inequalities that

$$\frac{d}{dt} \left| \nabla \left(e^{-F_{2t}t} b\right)^{\alpha+1} \right|^2 \leq C_{12} (|n|_{W^{1,\infty}(\Omega)}, |b|_{L^\infty(\Omega)}) \left( \left| \nabla \left(e^{-F_{2t}t} b\right)^{\alpha+1} \right|^2 + 1 \right).$$

Once again, we use Gronwall’s lemma and obtain

$$e^{-2(\alpha+1)F_{2t}t} \left| \nabla b^{\alpha+1}(t) \right|^2 \leq e^{C_{12} t} \left( |n(t)|_{W^{1,\infty}(\Omega)} |b(t)|_{L^\infty(\Omega)} \right) e^{C_{12} |n(t)|_{W^{1,\infty}(\Omega)} |b(t)|_{L^\infty(\Omega)}} d \left( \left| \nabla b^{\alpha+1}(t) \right|^2 + 1 \right) < \infty \text{ for all } t \in [0, T], T > 0.$$

Hence, $\left| b^{\alpha+1} \right|_{L^\infty((0, T) \times H^1(\Omega))} < \infty$ for all $T > 0$. 

\[\square\]
4. Uniqueness of solutions

In this section, we prove the uniqueness part of Theorem 2.6. As it is often the case for quasilinear systems with a non-monotonic principle part, a uniqueness result requires sufficient regularity of solutions. Our uniqueness result deals with regular weak global solutions introduced in Definition 2.4. The existence of such solutions is given by Theorem 2.6(1).

Further, as discussed in the introduction, equation (2.1b) for the $b$-component is not strongly parabolic and, more than that, it is not uniformly of porous medium-type near $t = 0$. This is the case for an initial value $n_0$ which is not strictly separated from 0. Consequently, the correct behaviour of $f_n$ and $f_s$ near the values $b = 0$ and $n = 0$ is required for uniqueness. It is given by conditions (2.10) - (2.11).

Proof of Theorem 2.6 (Uniqueness). Let the assumptions of the theorem hold and let $(n_1, b_1), (n_2, b_2)$ be two regular solutions (in terms of Definition 2.4) in $[0, T_0) \times \Omega$ of (2.1) to the same initial data $(n_0, b_0) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ with $b_0^{q+1} \in H^1(\Omega)$:

$$
n_1(0) = n_2(0) = n_0,
$$
$$
b_1(0) = b_2(0) = b_0.
$$

Our goal is to derive differential inequalities for some norms of the differences $n_1 - n_2$ and $b_1 - b_2$, from which uniqueness ultimately follows. The semilinear equation (2.1a) for the $n$-component allows to estimate $n_1 - n_2$ in different Sobolev spaces. The situation with $b_1 - b_2$, however, is different. Indeed, since equation (2.1b) is degenerate, a differential inequality for $b_1 - b_2$ can only be established for a rather weak norm. Already this requires an estimate for $n_1 - n_2$ in a stronger norm.

Let $T \in (0, T_0)$ be arbitrary. Due to Definition 2.4, the solutions $(n_1, b_1), (n_2, b_2)$ are bounded in the following sense:

$$
M := \max_{i=1,2} \left\{ ||n_i||_{L^\infty((0,T);W^{1,q}(\Omega))}, ||b_i||_{L^\infty((0,T);L^q(\Omega))}, ||b_0^{q+1}||_{L^q((0,T);H^1(\Omega))} \right\} < \infty.
$$

We will use this constant in our estimates.

Since both $(n_1, b_1)$ and $(n_2, b_2)$ are solutions of (2.1a), we get for their difference the equation

$$
\partial_t (n_1 - n_2) = d\Delta (n_1 - n_2) - (f_n(n_1, b_1) - f_n(n_2, b_2)).
$$

(4.1)

With the help of Assumption (2.10) on the function $f_n$, we obtain for $f_n(n_1, b_1) - f_n(n_2, b_2)$ the upper bound

$$
|f_n(n_1, b_1) - f_n(n_2, b_2)| \leq |f_n(n_1, b_1) - f_n(n_1, b_2)| + |f_n(n_1, b_2) - f_n(n_2, b_2)|
$$

$\leq C_1(M) \left( ||n_1 - n_2|| + ||n_1^\frac{q}{p} (b_1^{q+1} - b_2^{q+1})|| \right)$ in $[0, T]$. (4.2)

Multiplying (4.1) by $2(n_1 - n_2)$, we integrate by parts over $\Omega$ and obtain by means of (4.2) that

$$
\frac{d}{dt} ||n_1 - n_2||^2 = -2d_n ||\nabla (n_1 - n_2)||^2 + 2(f_n(n_1, b_1) - f_n(n_2, b_2), n_1 - n_2)
$$

$\leq C_2(M) \left( ||n_1 - n_2||^2 + ||n_1^\frac{q}{p} (b_1^{q+1} - b_2^{q+1})||^2 \right)$ in $[0, T]$.

Gronwall’s lemma then yields

$$
||n_1(t) - n_2(t)||^2 \leq C_3(M, T) \int_0^t e^{(t-\omega)C_2(M)} \left( ||n_1^\frac{q}{p} (b_1^{q+1} - b_2^{q+1})||^2 \right) d\omega
$$

$\leq C_3(M, T) \int_0^t ||n_1^\frac{q}{p} (b_1^{q+1} - b_2^{q+1})||^2 d\omega$ in $[0, T]$. (4.3)
Further, multiplying (4.1) by $-\Delta(n_1 - n_2)$, we integrate by parts over $[0, T] \times \Omega$, use the Cauchy-Schwarz and Young’s inequalities, as well as estimate (4.2), and get

$$
\frac{1}{2}\|\nabla(n_1 - n_2)(T)\|^2 = - d \int_0^T \|\Delta(n_1 - n_2)\|^2 \, d\omega + \int_0^T (f_\nu(n_1, b_1) - f_\nu(n_2, b_2), \Delta(n_1 - n_2)) \, d\omega
$$

$$
\leq - \frac{d}{2} \int_0^T \|\Delta(n_1 - n_2)\|^2 \, d\omega + C_4(M) \int_0^T \|n_1^\beta \left( b_1^{\alpha+1} - b_2^{\alpha+1} \right) \|^2 + \|n_1 - n_2\|^2 \, d\omega. \tag{4.4}
$$

Combining (4.3) and (4.4), we deduce that

$$
\|\nabla(n_1 - n_2)(T)\|^2_{H^1(\Omega)} + \int_0^T \|\Delta(n_1 - n_2)\|^2 + \|n_1 - n_2\|^2 \, d\omega \leq C_5(M, T) \int_0^T \|n_1^\beta \left( b_1^{\alpha+1} - b_2^{\alpha+1} \right) \|^2 \, d\omega. \tag{4.5}
$$

A Sobolev inequality for $N \leq 3$ and a maximal regularity result for elliptic equations (see, e.g., [16, Chapter 2, §5, Theorem 5.2]) allow us to conclude from (4.5) that

$$
\int_0^T \|n_1 - n_2\|^2_{H^2(\Omega)} \leq C_6 \int_0^T \|\Delta(n_1 - n_2)\|^2 + \|n_1 - n_2\|^2 \, d\omega
$$

$$
\leq C_7 \int_0^T \|\nabla(n_1 - n_2)\|^2 + \|n_1 - n_2\|^2 \, d\omega
$$

Next, we consider the difference $b_1 - b_2$. Due to Definition 2.4(iv), it satisfies for all test functions $\varphi \in H^1(\Omega)$ the following equation:

$$
(\partial_t(b_1 - b_2), \varphi) = - d_6 \left( n_1^\beta \nabla b_1^{\alpha+1} - n_2^\beta \nabla b_2^{\alpha+1}, \nabla \varphi \right) + (f_\beta(n_1, b_1) - f_\beta(n_2, b_2), \varphi) \text{ a.e. in } (0, T). \tag{4.7}
$$

To gain an estimate for $b_1 - b_2$, we use a specific test function $\varphi$ which is the solution of the following problem:

$$
\begin{cases}
\Delta \varphi = b_2 - b_1 & \text{in } [0, T] \times \Omega, \\
\partial_\nu \varphi = 0 & \text{on } [0, T] \times \partial \Omega, \\
\varphi_{n_1} = 0 & \text{on } [0, T].
\end{cases}
$$

Due to $b(t) \in L^2(\Omega)$ for all $t \in [0, T]$, such $\varphi$ has the required regularity: $\varphi(t) \in H^1(\Omega)$ for all $t \in [0, T]$, so that it is, indeed, an admissible test function. Integrating (4.7) by parts yields

$$
\frac{1}{2} \frac{d}{dt} \|\nabla \varphi\|^2 = - \frac{d_6}{\alpha + 1} \left( n_1^\beta (b_1^{\alpha+1} - b_2^{\alpha+1})(b_1 - b_2) \right) - \frac{d_6}{\alpha + 1} \left( n_2^\beta (b_1^{\alpha+1} - b_2^{\alpha+1})\nabla n_1^\beta, \nabla \varphi \right) - \frac{d_6}{\alpha + 1} \left( n_1^\beta - n_2^\beta \right) \nabla b_2^{\alpha+1}, \nabla \varphi
$$

$$
+ (f_\beta(n_1, b_1) - f_\beta(n_2, b_2), \varphi). \tag{4.8}
$$

With the assumption $\beta \geq 2$ and the inequality

$$
(b_1^{\alpha+1} - b_2^{\alpha+1})(b_1 - b_2) \geq C_9 \left( b_1^{\alpha+1} - b_2^{\alpha+1} \right)^2,
$$

we obtain from (4.8) the estimate

$$
\frac{1}{2} \frac{d}{dt} \|\nabla \varphi\|^2 \leq - C_{10} \left( n_1^\beta \left( b_1^{\alpha+1} - b_2^{\alpha+1} \right) \right)^2 + C_{11}(M, T) \left( n_1^\beta \left( b_1^{\alpha+1} - b_2^{\alpha+1} \right) \right)^2 \|\nabla \varphi\| + C_{12}(M, T) \|n_1 - n_2\|_{L^2} \|\nabla \varphi\|
$$

$$
+ F_{22} \|\nabla \varphi\|^2 + ||(f_\delta(n_1, b_1) - F_{22}b) - (f_\delta(n_2, b_2) - F_{22}b)|| \|\varphi\|. \tag{4.9}
$$
Integrating (4.9) over \([0, t]\) for \(t \in (0, T)\), we get by Poincaré’s and Young’s inequalities, the estimate (4.6) and an estimate for \(f_b(n, b) - F_{22} b\) similar to (4.2) that

\[
\|\nabla \varphi(t)\|^2 + \int_0^t \left( \varepsilon_1^{\frac{\alpha_2}{2}} \left( b_1^{\frac{\alpha_2}{2} + 1} - b_2^{\frac{\alpha_2}{2} + 1} \right) \right)^2 d\omega \leq C_{13}(M, T) \int_0^t \|\nabla \varphi\|^2 d\omega.
\]

Finally, Gronwall’s lemma yields

\[
\|\nabla \varphi(t)\| = 0 \text{ for all } t \in [0, T],
\]

and, thus, \(b_2 - b_1 = \Delta \varphi = 0\) almost everywhere in \([0, T] \times \Omega\). From one of the estimates for \(n_1 - n_2\), e.g., (4.3) we then conclude that \(n_1 - n_2 = 0\) almost everywhere in \([0, T] \times \Omega\). Since \(T \in (0, T_\bullet)\) was arbitrary, this proves uniqueness.

\[
\square
\]

5. Numerical simulations

In this section, we study the model (2.1) numerically in a two-dimensional spatial domain \(
\Omega := (-100, 100)^2 \)
. Our goal here is to illustrate the difference in pattern formation in the situations when the growth condition (2.8) is fulfilled and when it is violated.

For our simulations, we choose the shape of the reaction terms in accordance with Example 2.7, case 1 or 2. We consider for both cases several parameter values for the governing parameters in the diffusion and reaction terms. Since a subcase of Example 2.7 with parameters \(\alpha = \beta = 1\) and \(a_1 = a_2 = c = 0\) corresponds to the original model from [3], we choose the parameters \(d_n, d_b, G_1, G_2\) and the initial data \(n_0, b_0\) as in [3], in order to ensure that we can obtain similar patterns. The values which are kept constant throughout our calculations can be found in Table 1.

Remark 5.1 It has to be noted at this point that the above conditions, under which the simulations are performed, do not strictly fulfil the basic assumptions, under which the analysis of the preceding Sections 3 and 4 was carried out. Indeed:

1. The constant \(d_b\) was assumed to be constant, whereas, in our simulations, we follow [3] and consider the non-constant coefficient of the form

   \[ d_b = d_B (1 + \Delta \rho), \]

   where \(\Delta \rho\) is a random variable from a triangular distribution on \([-\rho, \rho]\) with \(0 < \rho < 1\). This modification accounts for the slight random movement of the bacteria on the agar plate and leads to the loss of symmetry in the pattern. The difference can be seen in Figure 1.

2. The spatial domain of the simulation is a square, so that it does not belong to the class \(C^2\). However, since we are dealing with the case of homogeneous Neumann boundary conditions, the irregularity of the domain is of little issue in the early stages of a pattern formation which takes place far away from the boundary of the domain.

<table>
<thead>
<tr>
<th>nutrient diffusion</th>
<th>bacteria diffusion</th>
<th>consumption rate</th>
<th>growth rate</th>
<th>initial nutrient</th>
<th>initial bacteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d_n)</td>
<td>(d_B)</td>
<td>(\rho)</td>
<td>(G_1)</td>
<td>(G_2)</td>
<td>(n_0)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.01</td>
<td>1</td>
<td>1</td>
<td>0.71</td>
</tr>
</tbody>
</table>

Table 1. Parameter values used for the simulations.

For our simulations, we use an adaptive implicit finite element method. This means that our discretisation is implicit in time and our space discretisation is a conforming finite element method. Further to speed up our calculations, we use adaptivity in time and space. The space adaptivity is regulated by a gradient method and a Dörfler marking [17], while the time adaptivity is determined...
by the number of Newton iterations and an error estimator. This method is implemented in FEniCS [18], a Python based Finite Element toolbox and the visualisation is done with Paraview [19].

As first model for our simulations, we take Example 2.7, case 1. Under this specific choice of the parameters, condition (2.8) can be simplified (see (2.13)) to \( a \geq \beta \geq 2 \).

In Figure 2, we illustrate the parameters we use for six different simulations and the admissible sets. Area (1) corresponds to condition (2.8) and area (2) corresponds to the assumption given by \( \beta \geq 2 \). The two points (marked by squares) in area (1) correspond to the circular patterns in Figure 3, while the two points closest to area (1) (marked by circles) correspond to the slight violation of (2.8), which results in the fingerlike patterns without blow ups. Finally, the remaining two points (marked by diamonds) violate (2.8) more severely and correspond to the two patterns with blow ups in Figure 5.

The simulation result of one parameter set which fulfills this condition can be seen in Figure 3. Even though we use a random \( d_b \), we obtain a perfect symmetry. A reason for this is that we have high parameter values which affect the growth term, such that the small random fluctuation can be neglected.

The results obtained when condition (2.13) is not fulfilled can be seen in Figures 4 and 5. One observes two different results, depending on how badly condition (2.13) is violated. If \( a \) is not much smaller than \( \beta \), we obtain a pattern with fingers as it is seen...
and reproduced by the model in [3]. The pattern is even smooth at the moving front, see Figure 4. If, however, $a$ is much smaller than $\beta$, then, as seen in Figure 5, high instabilities and blow ups appear at the moving front. Figure 3 and 5 include both the bacterial patterns and the nutrient concentration $n$ (on the right). As expected, the finite-time blow-ups of the bacterial density observed in the patterns in Figure 5 correspond to the degeneration of the variable $n$ (i.e., $n \to 0$). We observe numerically a strong blow up in $b$, while the nutrient concentration is still reasonably bounded.

For our second case in Example 2.7, we satisfy (2.8), which can be simplified under the assumptions $\alpha := \beta := 1, a_1 := a_2 := a$ to $\frac{c+1}{2-c} \leq a$. As it was observed in the simulations before, if this condition is fulfilled, the bacteria produce a round pattern, see Figure 6. The reason for the round patterns is the same as in the first case. The difference between the two patterns is that the right pattern has a higher growth parameter. Thus the pattern is rounder, and there is a higher concentration of bacteria.

The result when this condition is violated can be seen in Figure 7. The more we violate (2.14), the more fingers we obtain in the pattern. In this special case, we were unable to obtain instabilities like those we have seen in Figure 4 for case 1.

Figure 4. (2.13) not fulfilled: $\alpha = 2, \beta = 2, a = 1$ (left), $\alpha = 3, \beta = 2, a = 1$ (middle) and $\alpha = 2, \beta = 1, a = 0$ (right).

Figure 5. (2.13) not fulfilled: $\alpha = 2, \beta = 2, a = 0$ bacteria (left) and $\alpha = 3, \beta = 3, a = 0$ bacteria (middle) and nutrients (right) for $\alpha = 3, \beta = 3, a = 0$.

Figure 6. (2.14) fulfilled: $a = 2, c = 1$

Figure 7. (2.14) not fulfilled: $a = 1, c = 1$ (left), $a = 0, c = 2$ (mid) and $a = 0, c = 1$ (right).
Remark 5.2 Our simulations show that violation of condition (2.8) is essential for the non-trivial finger-like pattern formation. Indeed, we have essentially three cases:

1. The condition (2.8) is fulfilled. The diffusion of the growing population is strong in all directions, thus leading to a homogeneous pattern (e.g., as seen in Figures 3 and 6), with no blow-ups in finite time.

2. The condition (2.8) is violated slightly. The moving front of the pattern is smooth, but no more homogeneous: ‘fingers’ as in Figures 4 and 7 begin to form.

3. The condition (2.8) is severely violated. High instabilities and finite-time blowups take place (see, e.g., Figure 5).

Thus, the condition (2.8) can be used as a “weak guide line” for the respective pattern we expect/want to obtain.

6. Conclusion

In this article, we introduced and studied, both analytically and numerically, a model prototype for the pattern formation of a bacterial community growing and spreading in the presence of nutrients on an experimental agar plate. Our modelling includes different parameters that characterise the bacteria, the availability of nutrient, the thickness of the agar etc. Firstly, we obtain analytically rather strong conditions under which we can show existence and uniqueness. Secondly, we show numerically that the growth condition (2.8) plays an important role in the pattern formation.

References