Hybrid-mixed discretization of elasto-dynamic contact problems using consistent singular mass matrices

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SUMMARY

An alternative spatial semi-discretization of dynamic contact based on a modified Hamilton’s principle is proposed. The modified Hamilton’s principle uses displacement, velocity and momentum as variables, which allows their independent spatial discretization. Along with a local static condensation for velocity and momentum it leads to an approach with a hybrid-mixed consistent mass matrix (HMCMM). An attractive feature of such a formulation is the possibility to construct hybrid singular mass matrices (HSMM) with zero components at those nodes where contact is collocated. This improves numerical stability of the semi-discrete problem: the differential index of the underlying differential-algebraic system is reduced from 3 to 1 and spurious oscillations in the contact pressure, which are commonly reported for formulations with Lagrange multipliers, are significantly reduced. Results of numerical experiments for truss and Timoshenko beam elements are discussed. In addition, the properties of the novel discretization scheme for an unconstrained dynamic problem are assessed by a dispersion analysis. Copyright © 2010 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Standard spatial discretization of dynamic contact problems using Lagrange multipliers leads to systems of differential algebraic equations (DAE) with differential index 3 (see [11])

\[
\begin{align*}
\mathbf{M} \ddot{\mathbf{U}} + \mathbf{f}^{int}(\mathbf{U}, \dot{\mathbf{U}}, t) &= \mathbf{f}^{ext}(t) + \mathbf{G}(\mathbf{U}, t)\mathbf{Z} \\
\mathbf{G}^T \mathbf{U} - g_0 &\geq \mathbf{0} \quad \mathbf{Z} \leq \mathbf{0} \quad \mathbf{Z} (\mathbf{G}^T \mathbf{U} - g_0) = \mathbf{0}.
\end{align*}
\]

We note that the non-penetration and complementary conditions given by (1b) do not act on all nodes but only on a proper subset. As a result, we can decompose \(\mathbf{U}^T = \begin{pmatrix} \mathbf{U}^T \mathbf{c} \end{pmatrix} \) and find for \(\mathbf{G} = \begin{pmatrix} \mathbf{0} \\ \mathbf{G}^c \end{pmatrix} \). The numerical solution of such a system is challenging. Standard time integration schemes for ordinary differential equations (ODE) applied to System (1) may produce oscillating Lagrange multipliers and a substantial loss or gain in the total energy [1, 11]. This type of spurious oscillation does not occur or is significantly reduced in case of DAE systems with smaller index. This observation motivates the combination of space and time discretization with the goal to reduce the index.

There are three principal approaches to reduce the index. The first is a modification of the algebraic constraints to a differential equation, e.g., Baumgarte stabilization 1-index form [13] and Gear-Gupta-Leimkuhler 2-index form [12]. In the second approach, System (1) is solved directly in terms of predictor-corrector methods (see [10, 22] for references). The third family makes use of a
singular mass matrix $M^*$. In the case of linear elasto-dynamic systems, (1) can thus be reduced to

$$
\begin{bmatrix}
M^i & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{U}^i \\
\dot{U}^c
\end{bmatrix}
+ \begin{bmatrix}
K^{ii} & K^{ic} \\
K^{ci} & K^{cc}
\end{bmatrix}
\begin{bmatrix}
U^i \\
U^c
\end{bmatrix}
= \begin{bmatrix}
f^i \\
f^c
\end{bmatrix}
+ \begin{bmatrix}
0 \\
G^c
\end{bmatrix} Z
$$

(2a)

$$(G^c)^T U^c - g_0 \geq 0 \quad Z \leq 0 \quad Z \left((G^c)^T U^c - g_0\right) = 0. \quad (2b)$$

with $U^i$ and $U^c$ being the inner nodes and contact nodes, respectively.

As all equations that include the Lagrange multiplier $Z$ are now algebraic, the index of the system reduces to 1. This is a much more suitable problem from the point of view of time integration. We follow this approach in this paper, which has been originally introduced for one-side contact problems in [14].

In the literature there do exist two different approaches to construct a singular mass matrix $M^*$. Both preserve the total mass and work with standard volumetric finite elements. The first approach can be found in [9, 14] and redistributes the mass globally by solving a $L^2$ minimization problem. The second approach is proposed and analyzed in [19, 2] and it based on inexact quadrature formulas to assemble the entries of the mass matrix. However both approaches cannot be applied to thin-walled structures. Thus we focus on an alternative hybrid-mixed space discretization with different ansatz spaces for displacement and velocity [15, 17, 27].

Hybrid-mixed formulations have been applied also for membranes in elasto-dynamic contact [9] and for the eigenvalue analysis of plates [17] and can be extended to other types of thin-walled structures. Instead of constructing a weak form using the Galerkin method, as it was proposed in [9, 15], we use two- and three-field variational principles of elasto-dynamics (cf. [4, 17]). Three-field approach provide a very powerful and flexible framework for the construction of robust discretizations. In the case of nearly incompressible linear elasticity, volumetric locking can be avoided. A spatial discretization of the mixed formulation produce additional unknowns for the velocity and the momentum. A local static condensation for the velocity and the momentum leads to an approach with a hybrid-mixed consistent mass matrix (HMCMM). The choice of a poorer
approximation space for the velocity leads to a singular HMCMM, which is denoted as HSMM. If additionally the displacement shape function are re-parametrized, then HSMM has zero rows and columns at contact nodes, which is required in System (2). In other words, the application of a different space discretization provides a better semi-discretized problem (DAE index 1). Integration in time of the obtained semi-discrete problem then requires less elaborate methods. It can even be accomplished with standard ODE solvers, see [11].

In this paper, we present first a general formulation of elasto-dynamic contact based on a modified Hamilton’s principle. Secondly, we illustrate the efficiency of the formulation on the example of a 1D truss and a 2D Timoshenko beam in linear elasto-dynamics. As our primary motivation is the solution of impact problems, where wave propagation plays an important role, we also examine the proposed elements using a dispersion analysis.

The paper is organized as follows. In Section 2, the local form of an elasto-dynamic problem of a 2D thin-walled structure is given. The standard hyperbolic variational inequality for the problem is also provided. Section 3 is devoted to the modified Hamilton’s principle and its extension to a small sliding unilateral contact problem. We build a formulation that can be directly discretized. In addition, the equivalence of the modified Hamilton’s principle in contact to the local formulation is shown by derivation of the Euler-Lagrange equations. In Section 4, the HMCMM is discussed, and its stability is considered. Moreover, we derive the condition for the construction of a HSMM with zero components at certain nodes. The latter condition requires special shape functions for the displacements. In Section 5, we build such functions for arbitrary line elements. The newly developed mass matrix is rigorously analyzed for three important applications, such as free vibrations, wave propagation and impact problems. The quality of the solution for the modal analysis is indirectly assessed using dispersion analysis and illustrate the theoretical results. The results of such an analysis for the elements are presented in Section 6. Numerical examples for wave propagation and impact problems are presented in Section 6. Finally, the main results and the features of the new method and an outlook is given in Section 7.
2. PROBLEM STATEMENT

Let $B \subset \mathbb{R}^2$ be a reference mid-line of a 2D thin-walled body (Figure 1). $\partial B_u$ refers to the point where displacement boundary conditions are prescribed, and $\partial B_c$ denotes the free end point of the mid-line. Consider the following linearized elasto-dynamic contact problem of the body with a rigid obstacle

$$\begin{cases}
\rho \ddot{u} = L^* \sigma(u) + \hat{b} + t_c & \text{in } (0, T] \times B \\
u = 0 & \text{in } (0, T] \times \partial B_u \\
\sigma = \hat{t} + t_f & \text{in } (0, T] \times \partial B_c \\
u(0, .) = u_0 & \text{in } B \\
\dot{u}(0, .) = v_0 & \text{in } B
\end{cases}$$

(3)

where $u(t, x) : \mathbb{R}^2 \rightarrow \mathbb{R}^{dim}$ is the generalized displacement vector containing displacement and, possibly, rotational degrees of freedom, and $t_c$ and $t_f$ are the contact tractions in domain $B$ and at the end point $\partial B_c$. In equation 3 $u$, $t_c$ and $t_f$ are unknowns. $\rho$ is the tensor of inertia properties reduced to the mid-line, $\hat{b}$ and $\hat{t}$ are length-specific body force and traction at end points, respectively. The stress $\sigma$ is given by pre-integrated constitutive equations

$$\sigma = D \varepsilon, \quad \varepsilon = Lu$$

(4)

with the generalized strain vector $\varepsilon$. $D$ stands for the material matrix obtained by through-the-thickness pre-integration, and $L$ is the differential operator depending on the kinematics of the underlying truss or beam theory. Note, that the differential operator $L^*$ is usually adjoint to $L$. In addition, we allow $\rho$ and $D$ to be non-constant over the domain $B$. For more details concerning structural mechanics of thin-walled structures in matrix/operator format see [5, pp. 286-290].

We restrict ourselves to the case of frictionless unilateral contact and to the case where contact inside the domain appears only from one side of the mid-line. Thus, contact conditions can be
written as

\[ u \cdot n \leq g_B \quad t_c(u \cdot n - g_B) = 0 \quad t_c \leq 0 \quad \text{in } B \quad (5a) \]

\[ u \cdot \tau \leq g_{\Gamma} \quad t_{\Gamma}(u \cdot \tau - g_{\Gamma}) = 0 \quad t_{\Gamma} \leq 0 \quad \text{in } \partial B_c \quad (5b) \]

with initial gaps \( g_B \) and \( g_{\Gamma} \) as well as tangent and normal vectors \( \tau \) and \( n = e_3 \times \tau \), respectively.

Equations (3-5) define a closed system of differential-algebraic equations in partial derivatives.

Next, the corresponding weak form in space is introduced. Vector-valued spaces are denoted by bold letters, e.g., \( L^2(B) = [L^2(B)]^{dim} \). We define \( V := \{ u \in H^1 : u = 0 \text{ in } \partial B_u \} \) as test space for the displacements and a convex subset of \( V \) that satisfies the kinematic contact constraints \( K := \{ u \in V : u \cdot n \leq g_B \text{ in } B \text{ and } u \cdot \tau \leq g_{\Gamma} \text{ in } \partial B_c \} \). Finally, we arrive at the following variational inequality with respect to \( u \in L^2((0, T], K) \):

\[
\begin{align*}
& m(\ddot{u}, \delta u) + a(u, \delta u) \geq f(\delta u), \quad \forall \delta u = z - u, \quad t \in (0, T) \\
& (u(0, .) - u_0, w) = 0 \\
& (\dot{u}(0, .) - v_0, w) = 0
\end{align*}
\]

for \( z \in K, w \in V, \dot{u} \in L^2((0, T], L^2) \) and \( \ddot{u} \in L^2((0, T], V') \). The involved bilinear forms and linear forms are defined as \( m(\ddot{u}, \delta u) = \int_B \ddot{u} \cdot \rho \delta u \, dB, \quad a(u, \delta u) = \int_B \varepsilon(u) \cdot D \varepsilon(\delta u) \, dB, \quad (u, w) = \int_B u \cdot w \, dB \) and \( f(\delta u) = \int_B \hat{b} \cdot \delta u \, dB + (\hat{t} \cdot \delta u)|_{\partial B_c} \).

A purely displacement based discretization of the variational inequality [6, 7] leads to a standard DAE with differential index 3.

3. AN ALTERNATIVE THREE-FIELD MIXED FORMULATION

Omitting the contact constraints for the time being, i.e., \( K = V \), the Hamilton’s principle for the remaining system states that among all admissible displacements which satisfy the prescribed geometrical boundary conditions and the prescribed conditions at the time limits \( t = 0 \) and \( t = T \)
the actual solution makes the following functional stationary [4, p. 105]

\[ H(u) = \int_0^T \left( \frac{1}{2} m(\dot{u}, \dot{u}) - \frac{1}{2} a(u, u) + f(u) \right) \, dt \rightarrow \text{stat.} \]  

(7)

On this basis it is possible to construct a modified Hamilton’s principle with relaxed requirements. Namely, the velocity field \( v \) may be introduced as independent variable and the kinematic relation can be satisfied as additional condition using a Lagrange multiplier field \( p \) [4, Appendix I, p. 375]

\[ H(u, v, p) = \int_0^T \left( \frac{1}{2} m(v, v) - (p, v - \dot{u}) - \frac{1}{2} a(u, u) + f(u) \right) \, dt \rightarrow \text{stat.} \]  

(8)

The contact conditions can be introduced to (8) in different ways. We can look for \( u \in K \) or put the constraint into the functional using an indicator function \( I_K(u) \) for the convex set \( K \) [16]

\[
I_B(u, x) = \begin{cases} 
\infty & \text{if } u \cdot n > g_B \ x \in B \\
0 & \text{otherwise}
\end{cases}, \quad (9a)
\]

\[
I_\Gamma(u, x) = \begin{cases} 
\infty & \text{if } u \cdot \tau > g_\Gamma \ x \in \partial B_c \\
0 & \text{otherwise}
\end{cases}, \quad (9b)
\]

\[
I_K(u) = \int_B I_B(u, x) \, dB + I_\Gamma(u, x)|_{\partial B_c}, \quad (9c)
\]

with \( I_B(u, x) \) and \( I_\Gamma(u, x) \) being indicator functions for the contact constraints inside the domain and the boundary, respectively. Thus, we arrive at the following space-time weak form of the problem

\[ H(u, v, p) = \int_0^T \left( \frac{1}{2} m(v, v) - (p, v - \dot{u}) - \frac{1}{2} a(u, u) + f(u) + I_K(u) \right) \, dt \rightarrow \text{stat}, \quad (10)\]
with \( u \in L^2((0, T], \mathcal{V}), v \in L^2((0, T], L_2), p \in L^2((0, T], L_2), \dot{u} \in L^2((0, T], L_2) \). Variation of (10) gives

\[
\delta H(u, v, p) = \int_0^T (m(v, \delta v) - (\delta p, v - \dot{u}) - (p, \delta v - \delta \dot{u}) - a(u, \delta u) + f(\delta u) + \delta I_K(u)) \, dt.
\]

If we additionally assume \( \dot{p} \in L^2((0, T], \mathcal{V}') \) and integrate by parts \( (p, \delta \dot{u}) \), we get

\[
\delta H(u, v, p) = \int_0^T ((\delta p, \dot{u} - v) - (p - \rho v, \delta v) - (\dot{p}, \delta u) - a(u, \delta u) + f(\delta u) + \delta I_K(u)) \, dt.
\]

This formulation is the starting point for the method proposed herein.

\textbf{Remark 1}

The Euler-Lagrange equations of the weak form (12) are a system of equations

\[
\begin{cases}
-\dot{p} + L^* \sigma(u) + \dot{b} \in N_B(u) & \text{in } (0, T] \times B \\
\sigma - \dot{t} \in N_\Gamma(u) & \text{in } (0, T] \times \partial B_c \\
\dot{u} = v & \text{in } (0, T] \times B \\
p = \rho v & \text{in } (0, T] \times B
\end{cases}
\]

with \( N_B(u) \) and \( N_\Gamma(u) \) being subdifferentials of the indicator functions \( I_B(u, x) \) and \( I_\Gamma(u, x) \) respectively and \( \in \) denotes the inclusion. Thus, from (13) \( p \) can be identified as a momentum field. Also using the properties of differential inclusion, it can be shown that (13) is equivalent to (3\text{1–3}). The initial condition (3\text{4–5}) can be introduced in by correct boundary terms \( BT \) at the time limits \( t = 0 \) and \( t = T \), e.g., following paper [28]

\[
\int_0^T \delta \left( \frac{1}{2} m(v, v) - (p, v - \dot{u}) - \frac{1}{2} a(u, u) + f(u) + I_K(u) \right) \, dt = BT
\]

\[
BT = (\delta p|_0, u|_0 - u_0) + (\delta u|_0, \rho v|_0) - (\delta u, p)|_T.
\]
The first term in $BT$ weakly impose the displacement initial condition while the second and third term are responsible for the velocity initial condition. However, this term $BT$ leads to a non-integrable differential in contrast to the modified Hamilton’s principle (1). Such treatment of the initial conditions is necessary for the finite element discretization in time. Here the mid-point rule is used for the time integration and the initial conditions are merely interpolated at nodes.

Remark 2

The weak form (12) relaxes the requirements implied on $u$. Thus, there is a higher flexibility in the choice of numerical schemes applied to (13). Moreover, it is likely that discontinuities in the velocity may be represented more efficiently.

Remark 3

Although (13) is a three-field formulation, using $p = \rho v$ gives rise to a two-field formulation [4, Appendix I, p. 375]

$$
\delta H(u, v) = \int_0^T \left( -m(\delta v, v - \dot{u}) - m(\dot{v}, \delta u) - a(u, \delta u) + f(\delta u) + \delta I_K(u) \right) dt \rightarrow \text{stat}. \quad (14)
$$

Note, that this formulation is different from the one proposed by Hauret [15], where the coupling of the velocities $v$ and the time derivative $\dot{u}$ is done with an abstract Lagrange multiplier field.

Remark 4

The weak forms (12) and (14) are derived from natural variational principles. It is shown later that the mass matrices obtained from these weak forms are symmetric by construction even for non-uniform $\rho$.

4. SPATIAL DISCRETIZATION OF INERTIAL TERMS

4.1. Derivation of the mass matrix

In a first step, the expression for the consistent mass matrix (CMM) is derived from the spatial discretization of the inertial term in the weak form (6). Note, that the same shape functions are used...
for displacements (and accelerations) and their variation

\[ \mathbf{u}_h = \mathbf{N} \mathbf{U}, \quad \delta \mathbf{u}_h = \mathbf{N} \delta \mathbf{U}, \quad \dot{\mathbf{u}}_h = \dot{\mathbf{N}} \dot{\mathbf{U}}, \]  

(15)

\[ \mathbf{M} = \int_{\mathcal{B}} \mathbf{N}^T \rho \mathbf{N} \, d\mathcal{B}. \]  

(16)

Here, \( \mathbf{U} \) is the vector of nodal displacements, \( \mathbf{N} \) is the shape function matrix.

In order to obtain a HMCMM from the weak form (12), the displacement, velocity and momentum fields, along with their variations are discretized as follows

\[ \mathbf{u}_h = \mathbf{N} \mathbf{U}, \quad \delta \mathbf{u}_h = \mathbf{N} \delta \mathbf{U}, \quad \dot{\mathbf{u}}_h = \dot{\mathbf{N}} \dot{\mathbf{U}}, \]  

(17a)

\[ \mathbf{v}_h = \Psi \mathbf{V}, \quad \delta \mathbf{v}_h = \Psi \delta \mathbf{V}, \]  

(17b)

\[ \mathbf{p}_h = \mathbf{X} \mathbf{P}, \quad \delta \mathbf{p}_h = \mathbf{X} \delta \mathbf{P}, \quad \dot{\mathbf{p}}_h = \mathbf{X} \dot{\mathbf{P}}. \]  

(17c)

\( \Psi \) and \( \mathbf{X} \) are shape functions for the velocity and the momentum field, \( \mathbf{V} \) and \( \mathbf{P} \) are coefficient vectors for the velocity \( \mathbf{v} \) and momentum \( \mathbf{p} \), respectively. Let’s denote by \( n_d \), \( n_v \) and \( n_p \) the dimensions of the vectors \( \mathbf{U}, \mathbf{V} \) and \( \mathbf{P} \), respectively. There is a certain flexibility in working with different basis functions and \( \mathbf{N}, \mathbf{X}, \Psi \) are not necessary equal. Substituting (17) in (12) leads to

\[ \delta H^{*,h} = \int_0^T \left( \left[ \delta \mathbf{V}^T (C \mathbf{V} - B^T \mathbf{P}) - \delta \mathbf{P}^T (B \mathbf{V} - A^T \dot{\mathbf{U}}) \right] - \dot{\mathbf{U}}^T (\dot{\mathbf{P}} + K \mathbf{U} - F^{ext} - Z) \right) \, dt \]  

(18)

with \( Z \) being the contact force vector depending on a particular discretization of \( I_K(\mathbf{u}) \). Herein, the Node-to-Segment algorithm is used with collocation only in the corner nodes of the elements. Furthermore, we define

\[ K = \int_{\mathcal{B}} (LN)^T D L N \, d\mathcal{B}, \quad F^{ext} = \int_{\mathcal{B}} N^T b \, d\mathcal{B}, \quad A = \int_{\mathcal{B}} N^T \mathbf{X} \, d\mathcal{B}, \quad B = \int_{\mathcal{B}} \mathbf{X}^T D L N \, d\mathcal{B}, \quad C = \int_{\mathcal{B}} \Psi^T \rho \Psi \, d\mathcal{B}. \]  

(19b)
in which the matrices $A$, $B$ and $C$ are independent on time, i.e., in the finite deformation (geometrically non-linear) case.

Taking into account that the variations of $\delta U$, $\delta V$ and $\delta P$ are independent, we obtain from (18) the following algebraic system

$$
\begin{cases}
B^T P = CV \\
A^T \dot{U} = BV \\
A\dot{P} + KU = F_{ext} + Z
\end{cases}
$$

which can be identified as discretized counterpart of the material, kinematic and kinetic equation, respectively. If the matrix $C$ is invertible, the velocity vector $V$ can be eliminated from (20$_1$) and (20$_2$) resulting in

$$
BC^{-1}B^T P = A^T \dot{U}.
$$

(21)

Remark 5

For linearly independent $\Psi$ and positive density $\rho$ the matrix $C$ is positive definite and invertible. The matrix product $BC^{-1}B^T$ is at least positive semi-definite, if $C$ is positive definite. It is positive definite if $B$ has full rank, and its number of rows is less or equal to the size of $C$, i.e.

$$
n_p \leq n_v.
$$

(22)

Let us assume that $B$ satisfies these two conditions, then $\overline{C} = (BC^{-1}B^T)^{-1}$ is well defined.

$$
\overline{C}A^T \ddot{U} = \dot{P}.
$$

(23)

Substitution of the latter in (20$_3$) allows to eliminate $P$ from the equation of motion

$$
A\overline{C}A^T \ddot{U} + KU = F_{ext} + Z.
$$

(24)
That delivers an expression for HMCMM

\[ M^* = A \bar{C} A^T = A (B C^{-1} B^T)^{-1} A^T. \]  \hfill (25)

The formula (25) can be illustrated with Figure 2. The kinetic energy in the modified Hamilton’s principle (10) is computed on the discrete velocity \( v^h \) producing the matrix \( C \). Then it is projected via the Lagrange multiplier \( p^h \) on \( \dot{u}^h \) in terms of the matrices \( A \) and \( B^T \) resulting in the matrix \( M^* \).

If the matrix \( B \) is square and non-singular, an auxiliary matrix \( T = A B^{-T} \) may be introduced to combine both projections. In this case, the expression for the matrix \( M^* \) simplifies to

\[ M^* = A B^{-T} C B^{-1} A^T = T C T^T. \]  \hfill (26)

It is identical to the transformation of a quadratic form \( C \) under linear transformation \( T^T : V \to \dot{U} \).

Remark 6

The variational index in the spatial dimensions of \( p \) and \( v \) in the weak form (12) is equal to 0, so inter-element continuity of the shape functions \( \Psi \) and \( \chi \) is not required for convergence. Thus, elimination of \( V \) and \( P \) can be done on the element level provided that the basis functions are element-wise supported. Then the global \( M^* \) can be assembled from the local element mass matrices, and \( C \) has a block diagonal structure [17].

Remark 7

The second condition for stability of this three-field method is that the intersection of the null-spaces of the mass and stiffness matrices must be empty (cf. p 371 [3] and [9, Theorem 1, condition (7)] for Hu-Washizu \( \sigma - \varepsilon - u \) elements)

\[ \text{Ker} K \cap \text{Ker} M^* = \emptyset. \]  \hfill (27)

This condition automatically provides that \( K + \beta M^* \) is not singular for any \( \beta > 0 \), and thus most time integration schemes are well-defined.
Remark 8

\( \mathbf{M^*} \) is at least semi-definite by construction. In order to impose 0 at the \( i^{th} \) diagonal component of \( \mathbf{M^*} \), \( \mathbf{A} \) and \( \mathbf{C} \) have to satisfy the condition

\[
\mathbf{M}^*_{ii} = e_i^T \mathbf{A} \mathbf{C} \mathbf{A}^T e_i = 0.
\] (28)

with \( e_i \) being the \( i^{th} \) unit vector. As \( \mathbf{C} \) is positive definite, this condition can hold only if \( \mathbf{A}^T e_i = 0 \), i.e.,

\[
\mathbf{A}^T e_i = \int_\mathcal{B} \mathbf{X}(\mathbf{Ne}_i) \, d\mathcal{B} = 0.
\] (29)

Automatically, with condition (29) the entire \( i^{th} \) row and column of the mass matrix \( \mathbf{M}^*_{ij} \) is 0

\[
\mathbf{M}^*_{ij} = e_i^T \mathbf{A} \mathbf{C} \mathbf{A}^T e_j = (e_i^T \mathbf{A}) \, \mathbf{C} \mathbf{A}^T e_j = 0.
\] (30)

The condition (28) automatically satisfies the inf-sup condition given in [9] that is sufficient for the index reduction of DAE system (1). But it restricts the mass matrix to fit into the form given in (2). Such an approach is more complicated because it requires specially constructed shape functions for the displacement field. The advantage of the proposed approach is that massless nodes and nodes with mass are split by the structure of the mass matrix, and they can be treated separately.

Remark 9

The mass matrix \( \mathbf{M^*} \) is computed from the product of \( \mathbf{C} \) and \( \mathbf{A} \). The rank of \( \mathbf{C} \) is \( n_p \), therefore the rank of \( \mathbf{M^*} \) is less or equal than \( n_p \). At the same time from condition (27) it follows that the rank of \( \mathbf{M^*} \) must be greater than or equal to the number of zero eigen-values of \( \mathbf{K} \), which is equal to the number of rigid body modes of the element \( n_{rbm} \). If we want to make the number of massless nodes to be \( n_{d0} \), then the rank of \( \mathbf{M^*} \) is less or equal to \( n_d - n_{d0} \). It does not make sense to take more than \( n_d - n_{d0} \) momentum shape functions, because \( \mathbf{A} \) in this case does not have full rank. Its columns are linearly dependent, which means we can use only those shape functions of \( \mathbf{X} \) that provide the linear independent columns. This delivers the same mass matrix with less computational
effort. Thus, the number of momentum shape functions \( n_p \) should satisfy the following conditions

\[
n_{rbm} \leq n_p \leq n_d - n_{d0}.
\] (31)

The accuracy of different types of singular mass matrices is investigated by a spectral analysis, see also [2, 17]. We evaluate the accuracy of the singular hybrid-mixed mass matrix \( M^* \) via a dispersion analysis [20, 24].

5. DERIVATION OF SHAPE FUNCTIONS IN 1D

Here we construct shape functions for 1D elements that satisfy the orthogonality condition (29) at the corner nodes where later contact conditions are collocated. We restrict ourselves to the case \( n_d - 2 = n_v = n_p \) which automatically satisfies the stability conditions (22) and (31). This choice is also optimal with respect to computational cost for the element-wise computation of the singular mass matrix.

5.1. Three-node element

Consider a three-node 1D element (\( \text{dim} = 1, n_d = 3 \)). The goal is to obtain massless left and right nodes. First, for a consistent approximation of the momentum and the kinetic energy at least constant shape functions are required for the velocity (see Figure 4). The momentum shape functions \( \mathcal{X} \) enter into the zero mass condition (29). In order to avoid complications in the case of a non-uniform Jacobian, we firstly modify the momentum shape functions \( \mathcal{X} \) by multiplication with the factor \( \frac{|J_0|}{|J|} \).

\[
\Psi_1 = 1, \quad \mathcal{X}_1 = \frac{|J_0|}{|J|}, \quad n_v = n_p = 1,
\] (32)

where \( |J| \) is the determinant of the Jacobian, and \( |J_0| \) is evaluated at the center of the element \( \xi = 0 \).

Secondly, we assume that the new displacement shape functions have the same linear hull as the standard quadratic shape functions and that each shape function is a linear combination with so far unknown coefficients \( \alpha_i \).
Lin[N_1, N_2, N_3] = Lin[1, \xi, \xi^2]. \quad (33)

Thirdly, we assume that the interpolation condition is satisfied only at corner nodes

\begin{align*}
N_1(-1) &= 1, \quad N_2(-1) = 0, \quad N_3(-1) = 0, \\
N_1(1) &= 0, \quad N_2(1) = 0, \quad N_3(1) = 1.
\end{align*} \quad (34)

Finally, satisfying condition (29) with \(N_1 = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2\) leads to a system of equations

\begin{align*}
N_1(-1) &= 1 \\
N_1(1) &= 0 \\
\int_{-1}^{1} N_1 \left|J_0/J\right| d\xi &= \int_{-1}^{1} N_1 \left|J_0/J\right| d\xi = 0.
\end{align*} \quad (35)

The solution is \(N_1 = -\frac{1}{4} - \frac{1}{2} \xi + \frac{3}{4} \xi^2\). Due to symmetry we have \(N_3 = -\frac{1}{4} + \frac{1}{2} \xi + \frac{3}{4} \xi^2\). There are only 2 conditions for the 3 coefficients to define \(N_2\). To obtain a unique \(N_2\), we additionally require that \(N_1, N_2, N_3\) form a partition of unity

\[N_2 = 1 - N_1 - N_3 = \frac{3}{2} \left(1 - \xi^2\right).\] \quad (36)

Summarizing the results, see Figures 3 and 4 for an illustration, we get

\[
\begin{cases}
n_d = 3 : & N_1 = -\frac{1}{4} - \frac{1}{2} \xi + \frac{3}{4} \xi^2 \\
 & N_2 = \frac{3}{2} \left(1 - \xi^2\right) \\
 & N_3 = -\frac{1}{4} + \frac{1}{2} \xi + \frac{3}{4} \xi^2.
\end{cases}
\]

\[
\begin{cases}
n_v = 1 : & \Psi_1 = 1 \\
n_p = 1 : & X_1 = \left|J_0/J\right|
\end{cases}
\]
5.2. General case ($n_d$-node element)

Consider a $1D$ element with arbitrary number of nodes $n_d > 2$. There is more than one possibility to construct shape functions that satisfy the interpolation condition at corner nodes and the orthogonality condition (29). Here an option with closed form expressions for the shape functions is presented. It uses Gauss-Lobatto quadrature points as node locations and partially follows ideas of construction of biorthogonal bases proposed in [25].

Let us denote as $\{\xi_i\}_{i=1}^{n_d}$ and $\{w_i\}_{i=1}^{n_d}$ locations and weights for Gauss-Lobatto quadrature of order $n_d$ for the interval $[-1; 1]$, see [26, p. 887]. The special feature of Gauss-Lobatto quadrature is that integration points include end points of the interval ($\xi_1 = -1$ and $\xi_{n_d} = 1$). The inner points are the roots of the derivative of the Legendre polynomial $P'_{n_d-1}(\xi) = 0$, and the weights are equal to $w_i = \frac{2}{n_d(n_d-1)(P'_{n_d-1}(\xi_i))^2}$.

Then we define two Lagrange bases using the quadrature points as interpolation points

\[ \bar{N}_j = \prod_{\substack{i=1 \atop j \neq i}}^{n_d} \frac{\xi - \xi_i}{\xi_j - \xi_i}, \quad j = 1, n_d \]  
\[ \bar{\chi}_j = \prod_{\substack{i=2 \atop j \neq i}}^{n_d-1} \frac{\xi - \xi_i}{\xi_j - \xi_i}, \quad j = 2, (n_d - 1). \]

$\bar{\chi}$ and $\bar{N}$ are complete polynomials of order $n_d - 2$ and $n_d$, respectively. Moreover, $\bar{\chi}$ and $\bar{N}$ satisfy the interpolation condition at $\{\xi_i\}_{i=2}^{(n_d-1)}$ and $\{\xi_i\}_{i=1}^{n_d}$, respectively.

The shape functions for momenta are defined as complete polynomials of order $n_d - 2$ multiplied with $|J_0|/|J|$ 

\[ X_i = |J_0|/|J| \bar{\chi}_{i+1}, \quad j = 2, (n_d - 1). \]  

(39)

The left corner shape function can be constructed as linear combination of $\bar{N}_i$

\[ N_1 = \bar{N}_1 - \sum_{i=2}^{n_d-1} a_i \bar{N}_i. \]

(40)
The orthogonality condition for $N_1$ and $\chi_k$ reads as follows

$$\int_{-1}^{1} N_1 \chi_k |J| \, d\xi = |J_0| \int_{-1}^{1} \left( N_1 - \sum_{i=2}^{n_d-1} a_i \bar{N}_i \right) \bar{\chi}_k \, d\xi, \quad k = 2, (n_d - 1). \quad (41)$$

The integrand is a polynomial of order $(2n_d - 4)$ and thus the integral can be evaluated exactly in term of Gauss-Lobatto quadrature formula with $n_d$ nodes. Next, the interpolation property of $\bar{\chi}$ and $\bar{N}$ is used to simplify the expression

$$\int_{-1}^{1} N_1 \chi_k |J| \, d\xi = |J_0| \sum_{j=1}^{n_d} \left[ \left( \bar{N}_1(\xi_j) - \sum_{i=2}^{n_d-1} a_i \bar{N}_i(\xi_j) \right) \bar{\chi}_k(\xi_j) \right] w_j = |J_0| (\bar{\chi}_k(-1) w_1 - a_k w_k), \quad k = 2, (n_d - 1). \quad (42)$$

Then the orthogonality relation (41) gives

$$a_k = \frac{w_1 \bar{\chi}_k(-1)}{w_k} = \frac{\bar{\chi}_k(-1)}{(P_{n_d-1}(\xi_k))^2}. \quad (43)$$

Due to symmetry the right corner shape function can be constructed as

$$N_{n_d} = \bar{N}_{n_d} - \sum_{i=2}^{n_d-1} b_i \bar{N}_i, \quad (44a)$$

$$b_i = \frac{w_{n_d} \bar{\chi}_i(1)}{w_i} = \frac{\bar{\chi}_i(1)}{(P_{n_d-1}(\xi_i))^2}. \quad (44b)$$

Note, that the corner shape functions do not satisfy the interpolation condition at inner nodes.

The inner shape functions are not uniquely defined. They may be constructed such that $\{N_i\}_{i=1}^{n_d}$ form a partition of unity

$$N_i = (1 + a_i + b_i) \bar{N}_i = \left( 1 + \frac{\bar{\chi}_i(-1) + \bar{\chi}_i(1)}{(P_{n_d-1}(\xi_i))^2} \right) \bar{N}_i, \quad i = 2, (n_d - 1). \quad (45)$$

**Remark 10**

The functions $\{\bar{N}_i\}_{i=1}^{n_d}$ form a complete polynomial basis, and it is easy to check that $\{N_i\}_{i=1}^{n_d}$ also
do. The basis \( \{ \tilde{N}_i \}_{i=1}^{n_d} \) is built from \( \{ N_i \}_{i=1}^{n_d} \) via a linear transformation \( T_N \), which has a nice block structure

\[
N_i = \sum_{j=1}^{n_d} T_{N,ij} \tilde{N}_j, \quad (46a)
\]

\[
T_N = \begin{pmatrix}
1 & -a_2 & \cdots & -a_{n_d-1} & 0 \\
0 & (1 + a_2 + b_2) & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & (1 + a_{n_d-1} + b_{n_d-1}) & 0 \\
0 & -b_2 & \cdots & -b_{n_d-1} & 1
\end{pmatrix}, \quad (46b)
\]

\[
\det(T_N) = \prod_{i=2}^{n_d-1} (1 + a_i + b_i). \quad (46c)
\]

The absolute values of \( a_i \) and \( b_i \) are less than 0.5, which implies that \( (1 + a_i + b_i) > 0 \). Thus, the determinant of the matrix \( T_N \) is non-zero and the transformation matrix is regular.

In the case \( n_d = 3 \), the functions are identical to (37). In the case \( n_d = 4 \), the basis reduces to (see Figure 5)

\[
\xi_{1,4} = \pm 1; \quad \xi_{2,3} = \pm \frac{1}{\sqrt{5}}
\]

\[
\begin{align*}
\text{n}_d = 4 &: \quad N_1 &= -\frac{1}{4} + \frac{3}{4} \xi + \frac{3}{4} \xi^2 - \frac{5}{4} \xi^3 \\
&\quad N_2 &= \frac{3}{4} + \frac{3\sqrt{5}}{4} \xi - \frac{3}{8} \xi^2 - \frac{3\sqrt{5}}{4} \xi^3 \\
&\quad N_3 &= \frac{3}{4} - \frac{3\sqrt{5}}{4} \xi - \frac{3}{8} \xi^2 + \frac{3\sqrt{5}}{4} \xi^3 \\
&\quad N_4 &= -\frac{1}{4} - \frac{3}{4} \xi + \frac{3}{4} \xi^2 + \frac{5}{4} \xi^3
\end{align*}
\quad (47)
\]

\[
\begin{align*}
\text{n}_v = 2 &: \quad \Psi_1 &= 1 \Psi_2 = \xi \\
\text{n}_p = 2 &: \quad \lambda_1 &= \frac{|J_0|}{2|J|} / (\sqrt{5} \xi + 1) \quad \lambda_2 = \frac{|J_0|}{2|J|} / (-\sqrt{5} \xi + 1)
\end{align*}
\]

This method can be easily extended for multiple dimensions in case of tensor product structure.

5.3. Discussed elements

The formulation and shape functions developed above enable us to construct finite elements for thin-walled structures that significantly reduce artificial oscillations of contact forces in dynamic
problems and show sufficient accuracy for modal and wave propagation analysis. For illustration we restrict ourselves to $1D$ truss and $2D$ Timoshenko beam elements. We also consider only formulations with equal number of velocity and momentum parameters $n_v = n_p$.

We study two different truss elements

- $Tr2-0$: a three-node truss with shape functions (37)
- $Tr3-1$: a four-node truss with shape functions (47).

In the denomination $(TrX-Y)$ for the elements $X$ and $Y$ are the order of the displacement and velocity interpolation, respectively. For these elements contact may occur at the end of domain $\partial B_c$, i.e., the left or right node.

For a beam element a lateral contact inside domain $B$ may occur. It is collocated at the corner nodes of elements herein. Only three-node Timoshenko beam elements are presented

- $Ti2-2-2-2$: standard Lagrange formulation with CMM (used for reference computations)
- $Ti2-2-0-0$: shape functions (37) for displacement and rotation
- $Ti2-2-0-1$: shape functions (37) for displacement and rotation, but angular velocity is linear
- $Ti2-2-0-2$: shape functions (37) for displacement and rotation, but angular velocity is quadratic
- $Ti2*-2-1-2$: linked interpolation (48), see discussion below.

In the element notation $(TiX-Y-Z-W)$ 4 numbers stand for the order of displacement, rotation, translational and angular velocity, respectively. The latter 4 elements have singular mass matrices with zero masses at corner nodes. They differ in the computation of the rotational inertia. $Ti2-2-0-1$ and $Ti2-2-0-2$ use a higher order ansatz for angular velocity. The linked interpolation $Ti2*-2-1-2$ allows the element to yield an exact stiffness matrix for static problems. The shape functions reads
as follows

\[
\begin{bmatrix}
-1/4 + 3/4 \xi + 3/4 \xi^2 - 5/4 \xi^3 & 0 \\
-5/8 \xi \left(1 - \xi^2\right) \xi & 1/2 - 1/2 \xi \\
3/2 - 3/2 \xi^2 & 0 \\
0 & 1 - \xi^2 \\
-1/4 - 3/4 \xi + 3/4 \xi^2 + 5/4 \xi^3 & 0 \\
-5/8 \xi \left(1 - \xi^2\right) \xi & 1/2 + 1/2 \xi \\
\end{bmatrix}
\]

Moreover, the corner displacement shape functions are orthogonal to linear velocities. The linear translational velocity \( \Psi_{22} \) allows exact computation of the polar inertia, which is beneficial for flexural dominated problems.

6. NUMERICAL EXPERIMENTS

Numerical tests are conducted using the implementation of the elements in the computer algebra package Maple. The midpoint rule with constant time-step is used for all transient simulations \([19, 18]\). Every element is checked for three types of experiments. In the first test, we focus on dispersion analysis. The second benchmark is a wave propagation test. Finally, as third experiment, consider an impact problem. The results for our new elements are compared with the analytical solution or with results from the standard mass matrix approach.

The overall accuracy of the inertia and stiffness discretization can also be evaluated using a Fourier analysis of an infinite lattice of equally-sized elements. The results of such an analysis provide an estimate of the error in the eigenfrequencies needed for free vibration problems, and it gives an exact dispersion relation relevant for wave propagation problems. It allows to obtain solutions in the form of traveling harmonic waves \([20, 24]\) of wavenumber \( k = 2\pi/\lambda \), wavelength \( \lambda \), phase velocity \( c \) and circular frequency \( \omega \). For given elements multiple solutions, or branches,
of the dispersion analysis are possible. They correspond to physically different types of waves or numerical artifacts. For the elements discussed here we distinguish longitudinal, shear and bending waves. Also high order elements can exhibit acoustic and optical branches [24]. The optical modes are also called mesh modes, and they are unwanted. Here, we compare the dispersion relations for individual branches with corresponding analytical expression of the corresponding continuum problem. The difference in their Taylor expansions is taken as main error measure. For convenience, the dispersion relation are presented in dimensionless quantities with wavenumber \( \kappa = k/l_e \) and frequency \( \Omega = \omega l_e/c \).

6.1. Three-node truss element: Tr2-0

First, we perform a dispersion analysis. The stiffness matrix and the HSMM read

\[
\begin{bmatrix}
4 & -6 & 2 \\
-6 & 12 & -6 \\
2 & -6 & 4
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

with \( l_e \) being the element length. The solution of the equation of motion is a harmonic wave, and the ansatz for it at the \( j^{th} \) node reads

\[
u_j(t) = U_c \exp(i(\kappa j/2 - \Omega c t/l_e)) \quad \text{for } j \text{ odd,} \quad (50a)
\]

\[
u_j(t) = U_m \exp(i(\kappa j/2 - \Omega c t/l_e)) \quad \text{for } j \text{ even}, \quad (50b)
\]

with \( U_c \) and \( U_m \) being the complex-valued amplitudes at the corner and middle nodes, respectively, and \( c = \sqrt{E/\rho} \) being the speed of sound.
Substitution of the ansatz in the equation of motion leads to a homogeneous system with the two unknowns $U_c$ and $U_m$

\[
\begin{pmatrix}
-\Omega^2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
-6 & 12 & -6 & 0 & 0 \\
2 & -6 & 8 & -6 & 2
\end{pmatrix}
\begin{pmatrix}
U_c e^{-i\kappa} \\
U_m \\
U_c \\
U_m e^{i\kappa} \\
U_c e^{i\kappa}
\end{pmatrix} = 0, \quad (51a)
\]

\[
\begin{pmatrix}
-6 e^{-i\kappa} - 6 & -\Omega^2 + 12 \\
4 \cos(\kappa) + 8 & -6 - 6 e^{i\kappa}
\end{pmatrix}
\begin{pmatrix}
U_c \\
U_m
\end{pmatrix} = 0. \quad (51b)
\]

A non-trivial solution of (51b) only exists if the determinant of the system is zero. It yields the characteristic equation, from which the dispersion relation is obtained

\[
\Omega^2 = \frac{6(1 - \cos(\kappa))}{\cos(\kappa) + 2}. \quad (52)
\]

In the case of a 1D problem, the analytical solution predicts non-dispersive wave propagation $\Omega_0 = \kappa$, i.e., a constant wave speed for all wave numbers (see Figure 6). The semi-discrete solutions are dispersive. The Taylor expansion of the dispersion relation (52) at $\kappa = 0$ is exact up to a 4th order corresponding expression for the continuum problem $\Omega_0^2 = \kappa^2$

\[
\Omega^2 = \kappa^2 + \frac{1}{12} \kappa^4 + \frac{1}{360} \kappa^6 + O(\kappa^8). \quad (53)
\]

Thus, the phase velocity for long waves ($\kappa \to 0$) and the values of the lowest eigenfrequencies converge uniformly to the exact one. Surprisingly, the dispersion relation of Tr2-0 coincides with the dispersion relation of a 2-node truss element with CMM [21]. Moreover, the dispersion relation of Tr3-1 (see below (54)) coincides with the dispersion relation of a 3-node truss element with CMM [21].
To illustrate the capabilities of the element for the wave propagation problem, we consider a problem with a sharp shock front. The initial conditions correspond to a rectangular wave packet moving from the left side of a finite truss to the right (see Figure 7). The length of the packet is 20\% of the total truss length. The length of the truss is large enough to avoid reflections within the simulation time. The analytical solution predicts that the particle velocity $v_0$ in the packet is preserved during the wave propagation. Snapshots of the velocity profile are presented in Figure 8. It shows reasonably small dispersion, while partially the dispersion stems from time integration.

Next, the presented finite element formulation is tested in a rigid wall impact problem. The setup for this test is analogous to the wave propagation test apart from the fact that the initial velocity is uniform in the truss. The contact condition is applied on the node at the right end. The exact analytical solution predicts a constant contact force $F_{c}^{\text{anal}} = v_0 A \sqrt{\rho E}$ and a total impact duration of $T_c = 2 l \sqrt{\frac{E}{\rho}}$. For the specified data, the numeric values are $F_{c}^{\text{anal}} = 10$ and $T_c = 0.02$. The time history of the computed contact force for different time step sizes is presented in Figure 9. The contact persist during the entire impact, and the computed duration of the impact is close to the theoretical one, which is not the case for the standard mass matrix approach.

The maximum overshoot of the contact force $t_\Gamma / F_c$ is around 1.8 in the range of the studied time-steps, and the overshoot reduces with increasing size of the time steps. The reason for that is as follows: time integration with larger time-steps $dt > 3 l_e / c$ cuts off the highest modes in the solution. The phase velocity for the higher modes possesses a higher error (see Figure 6). This shows up also in the history of the contact force, where spurious irregular oscillations in the second half of the impact are caused by reflected higher modes. It also means, that the resolution of space and time discretization should match. The choice of the time step $dt$ in the range from $2 l_e / c$ to $5 l_e / c$ provides the most accurate results in the contact force for this benchmark.

6.2. Four-node truss element: $\text{Tr3-1}$

The spectral analysis for the element $\text{Tr3-1}$ is similar to $\text{Tr2-0}$, and its details are omitted. The spectral analysis gives two branches for the dispersion relation. These branches may be classified as
acoustic and optical branches \cite{20} with the following expressions for dispersion

\begin{align}
\Omega_{aco}^2 &= 4 \left( \frac{13 + 2 \cos(\kappa) - R_1}{3 - \cos(\kappa)} \right) \\
\Omega_{opt}^2 &= 4 \left( \frac{13 + 2 \cos(\kappa) + R_1}{3 - \cos(\kappa)} \right) 
\end{align}

with \( R_1 = \sqrt{124 + 112 \cos(\kappa) - 11 \cos^2(\kappa)} \). The Taylor expansion of the acoustic branch \( \Omega_{aco}^2 \) is exact up to \( O(\kappa^6) \), i.e., the first eigenvalue would converge with \( 6^{th} \) order, see Figure 10.

In the wave propagation test, the velocity profiles are less dispersive than the ones obtained with Tr2-0. This is due to the higher accuracy of the dispersion curve (54), see Figures 10 and 11.

The rigid wall impact problem produces a maximum overshoot \( t_1/F_c \) of about 1.7 (Figure 12). As in the case of the quadratic element, contact persists during the entire impact, and the computed impact duration is close to the theoretical one.

6.3. Three-node Timoshenko beam element

In this section, four different formulations of three-node shear deformable beam elements are tested. For the stiffness matrix (19), we use the two-point Gauss quadrature to circumvent shear locking in the static problem.

The spectral analysis is quite technical in this case. The main results of the analysis are summarized in Table I below. For comparison, the exact dispersion relation for a Timoshenko beam with rectangular cross section and \( \nu = 0.0 \), no shear correction (i.e. \( GA_s = GA \)) is given and reads as follows

\begin{align}
\omega_{bend}^2 &= \kappa^4 - 1/4 \Lambda^2 \kappa^6 + \frac{11}{144} \Lambda^4 \kappa^8 - \frac{5}{192} \Lambda^6 \kappa^{10} + O(\kappa^{12}) , \\
\omega_{shear}^2 &= \frac{72}{\Lambda^4} + 18 \frac{\kappa^2}{\Lambda^2} - \kappa^4 + 1/4 \Lambda^2 \kappa^6 - \frac{11}{144} \Lambda^4 \kappa^8 + \frac{5}{192} \Lambda^6 \kappa^{10} + O(\kappa^{12}) .
\end{align}

\( \Lambda = \frac{1}{t_e} \) is the length-to-thickness ratio of an individual element. The dispersion curves for the case of shear-to-bending ratio \( \frac{\overline{EA}}{EI} = 100 \) are given in Figure 13.
The dispersion properties of the element may also be evaluated from the following test problem. Consider a semi-infinite beam with a free left end. A constant velocity $v_0$ is prescribed at the free end. We model this problem with the finite length beam taking it long enough to avoid spurious reflections from the right end (see Figure 14). A regular mesh with slenderness $\Lambda = 7.0$ is used.

The numerical solutions are compared with an analytical solution obtained in [8]. Figure 15 compares velocity profiles at a given time for the four different element formulations. We plot the velocity by connecting values at the midpoints of the elements, however the distribution for formulation $T_2^*-2-1-2$ is actually piecewise linear and discontinuous. The analytical solution is dispersive, i.e., shear and flexural wave propagate through the beam with a velocity depending on the wave number $k$. The maximum velocity of flexural waves is smaller than the maximum velocity of shear waves by the factor $\sqrt{EA/G_A} \approx 1.4$ [8]. Therefore, a sharp front is predicted between shear and flexural wave packages (see Figure 15 at the location $x_1/r_g \approx 3.5$ with $r_g = \sqrt{I/A}$ being the radius of gyration). Formulations $T_2-2-0-0$ and $T_2-2-0-1$ can follow the front much better than $T_2-2-0-2$ and $T_2^*-2-1-2$. The results for $T_2-2-0-2$ and $T_2^*-2-1-2$ are almost identical, because the difference in their mass matrices reduces with large slenderness. Furthermore, the height of the spurious oscillation of the velocity profile around the front is comparable with the one obtained in an identical wave propagation test for two-node Timoshenko beam elements [23].

The performance in the case of elasto-dynamic contact is tested using a bounce problem (Figure 16). We consider a beam hinged on both ends with a sine profile for the initial velocity. There is a rigid obstacle with initial gap $g_B$. We use a coarse mesh with 4 elements, such that contact occurs only in the middle node, which is a massless node. The time interval is big enough for one impact.

The numerical results for the contact force at the middle node are given in Figure 17. The results for the standard mass matrix consist of several impacts with duration in one time step each. In the case of HSMM, spurious oscillations of the contact force is eliminated. All formulations proposed herein provide almost similar results, because rotational inertia – which makes the difference between the formulations – are not very significant in this test problem. A similar behavior for the
impact of a beam with an obstacle is reported in [27], where small differences have been obtained for linear and constant velocity field approximation in the case of the Euler-Bernoulli beam model.

Although all numerical results of the newly proposed approach are at least as good as and quite often superior to results in the standard case, there is one drawback. The condition number of the time integration matrix of the mid-point rule \( \left( \frac{4}{dt^2} M^* + K \right) \) starts growing for very small time-steps \( dt < \frac{l_e}{100c} \). The reason for this is that the singular term \( \frac{4}{dt^2} M^* \) for such small time-step sizes dominates the term \( K \). However, a simple Jacobi preconditioner solves this problem.

7. CONCLUSIONS AND OUTLOOK

An alternative spatial discretization of elasto-dynamic problems is presented. It is based on the three-field modified Hamilton’s principle with independent fields of velocity, momenta and displacements, and it exploits the use of singular mass matrices. It allows to reduce the differential index of the semi-discretized system from three to one if the approximation spaces for velocity, momenta and displacements are specially constructed. It does not increase the total number of unknowns, because the discretization parameters for velocity and momenta are eliminated locally.

The main result of the paper is the proposed discretization of the three-field modified Hamilton’s principle for contact problems. It combines a novel expression for a hybrid singular mass matrix, a condition for vanishing of mass at certain nodes and a new class of polynomial shape functions. It was shown that a diagonal entry of the element mass matrix vanishes if the shape function belonging to the corresponding node is orthogonal to all momenta shape functions. New shape functions that fulfill the orthogonality condition have been proposed. The elements use Gauss-Lobatto quadrature points as node location (see also [25]) and fulfill the interpolation condition only at the corner (vertex) nodes. It has been verified that the discretization satisfies the conditions for computability and stability needed within the framework of a hybrid-mixed approach.

Some crucial differences distinguish this discretization from the one presented in [9, 15]: First, a natural variational principle is used, which guarantees symmetry and definiteness of the mass matrix. Moreover, three independent variables for velocity, momentum and displacement along with

newly developed re-parametrized shape functions are used. The method can be used for thin-walled and bulk structures, i.e., contact may be realized at massless nodes inside the domain and on its boundary [2, 19]. Finally, the accuracy of the new family of FE for modal problems is evaluated by a dispersion analysis.

The approach efficiently reduces spurious temporal oscillations of the contact pressure. Persistent contact was obtained for all considered benchmarks. There is still some overshoot of contact pressure but its magnitude is acceptable; most importantly, it stays finite for very small time steps \((dt \to 0)\), in contrast to formulations with a standard consistent mass matrix. The maximal contact pressure converges quickly to the analytical value for time steps \(dt \approx l_e/c\) and stays close to it for larger time steps.

One limitation of the concept of hybrid singular mass matrices is the need of implicit time integration. So further studies are necessary to check competitiveness and efficiency of the method for various contact-impact applications (crash, deep-drawing, robot dynamics, etc.). Another approach for time integration is an explicit/implicit scheme where all inner nodes can be still handled explicitly.

This method can be viewed as a regularization of dynamic contact problems. Like in a penalty method a certain stiffness is assigned between nodes with mass and locations where contact is collocated. In contrast to a penalty method, contact constraints are fulfilled exactly (no penetration allowed). As only the massless nodes are subject to contact constraints, the distance between "contact nodes" and "mass nodes" plays the role of springs in a penalty method (typically one half or one third of an element length). For example, in the case of impact of a one-dimensional three-node element contact occurs at the corner node, but inertia is condensed in the middle node. An attractive feature of this sort of regularization is that the equivalent penalty stiffness is automatically adjusted in case of lateral and edge contact to bending/shear and membrane stiffness of the element, respectively.

The hybrid mixed consistent and singular mass matrices developed herein are symmetric and positive semi-definite by construction. The mass matrix is computed at the beginning of the
simulation at a cost comparable to a standard consistent mass matrix and it does not depend on the active contact set. The method affects only the inertia term of the weak form and can be combined with any arbitrary formulation for stiffness calculation (EAS, ANS, hybrid-mixed, etc.). Contact can be treated with a standard node-to-segment approach, available in many codes, applied to the set of massless nodes.

One issue is that the proposed method requires at least quadratic shape functions and a re-parametrization of the shape functions is needed. Elements with quadratic shape functions are not very popular for strongly non-linear problems, because of lack of stability. Additionally, re-parametrized shape functions require some extra effort in pre and post-processing with standard tools.

For smooth problems the order of convergence of the proposed method for the lowest eigenfrequencies is lower than when using standard consistent and lumped mass matrices. This is due to the poorer function spaces used for velocity and momentum. However, the behavior of elements in wave propagation problems is acceptable.

The method can be easily extended to shells and membranes as well as \(2D\) and \(3D\) bulk structures. The derived shape functions may be extended by means of a tensor product structure. Well tuned ansatz spaces have already been found for \(2D\) 9-node and \(3D\) 27-node solid elements and will be presented in a subsequent paper. Another direction of current research is the relaxation of the orthogonality condition (28) such that it would allow singular mass matrices for standard Lagrange shape functions.

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REFERENCES


Figure 1. Geometry of the thin-walled structure and definition contact constraints
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Figure 7. Wave propagation benchmark for trusses

- \( EA = 10^4 \)
- \( \rho = 1 \)
- \( v_0 = 0.1 \)
- \( l = 1 \)
- \( dt = 2 \times 10^{-5} \)
- \( T_{end} = 0.005 \)
Figure 8. Velocity profiles at different time points for quadratic truss element with constant velocities, Tr2-0
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- $E = 10^4$
- $\nu = 0$
- $\rho = 1$
- $h = 0.1$
- $v_0 = 0.1$
- $l = 1$
- $dt = 1 \cdot 10^{-5}$
- $t_{end} = 1.4 \cdot 10^{-3}$
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Figure 17. Contact force in the middle of the beam for bounce benchmark (a) comparison with standard mass (b) only singular mass matrices
<table>
<thead>
<tr>
<th>Instance name</th>
<th>Description</th>
<th>Dispersion behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ti2-2-2-2</td>
<td>Standard shape functions for displacement and rotations</td>
<td>3 branches. Flexural and shear branches match the analytical expressions for continuum beam up to $O(\kappa^8)$ and $O(\kappa^6)$. 3rd branch is optical lying between flexural and shear branches.</td>
</tr>
<tr>
<td>Ti2-2-0-0</td>
<td>Constant ansatz for translational and angular velocities</td>
<td>2 branches. Flexural and shear branches match the analytical expressions for continuum beam up to $O(\kappa^6)$ and $O(\kappa^2)$ respectively.</td>
</tr>
<tr>
<td>Ti2-2-0-1</td>
<td>Constant ansatz for translational velocity and linear angular velocities</td>
<td>3 branches. Flexural and shear branches match the analytical expressions for continuum beam up to $O(\kappa^6)$ and $O(\kappa^2)$ respectively. 3rd branch is optical with a gap above the shear branch.</td>
</tr>
<tr>
<td>Ti2-2-0-2</td>
<td>Constant ansatz for translational velocity and quadratic angular velocities</td>
<td>3 branches. Flexural and shear branches match the analytical expressions for continuum beam up to $O(\kappa^6)$ and $O(\kappa^4)$ respectively. 3rd branch is optical lying between flexural and shear branches.</td>
</tr>
<tr>
<td>Ti2*-2-1-2</td>
<td>Linked interpolation formulation after (48)</td>
<td>3 branches. Flexural and shear branches match the analytical expressions for continuum beam up to $O(\kappa^8)$ and $O(\kappa^6)$ respectively. 3rd branch is continuation of the flexural branch without a gap.</td>
</tr>
</tbody>
</table>

Table I. Comparison of 5 Timoshenko beam element formulations