A MORTAR BASED CONTACT FORMULATION FOR NON-LINEAR DYNAMIC PROBLEMS USING DUAL LAGRANGE MULTIPLIERS

S. Hartmann*, S. Brunssen†, E. Ramm* and B. Wohlmuth†

*Institute of Structural Mechanics
University of Stuttgart, Pfaffenwaldring 7, D-70550 Stuttgart, Germany
e-mail: hartmann/ramm@ibb.uni-stuttgart.de

† Institute for Applied Analysis and Numerical Simulation
University of Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany
e-mail: brunssen/wohlmuth@ians.uni-stuttgart.de

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Summary. This work presents a large deformation contact formulation for non-linear dynamic problems. The formulation is based on the mortar method and uses dual spaces for the interpolation of the Lagrange multiplier. A local basis transformation in combination with a primal-dual active set strategy allows a nodal decoupling of the contact constraints.

1 INTRODUCTION

Many existing algorithms for the analysis of large deformation contact problems use a so-called node-to-segment approach to discretize the contact interface between dissimilar meshes. It is well known, that this discretization strategy may lead to problems like loose of convergence or jumps in the contact forces. Additionally it is popular to use penalty methods to satisfy the contact constraints. This necessitates a user defined penalty parameter, the choice of which is somehow arbitrary, problem dependent and might influence the accuracy of the analysis. In this work, a segment-to-segment contact formulation is presented, that does not require any user defined parameter to handle the non-linearity of the contact conditions. The approach is based on the mortar method, enforcing the compatibility condition along the contact interface in a weak integral sense.

2 PROBLEM DESCRIPTION

A two body large deformation contact problem is given in Figure 1. The reference configurations of the contacting bodies are represented by Ω(1) and Ω(2). With u(α) (α = 1, 2) the time dependent displacement fields are denoted and the potential contact surfaces are identified with Γ(α)c and γ(α)c in the reference and current configuration, respectively. A scalar valued gap function g = −ν(x) · (x − ˆy) is defined, where ν(x) denotes the outward unit normal to γ(1)c at x and ˆy is an appropriate projection point on γ(2)c. Applying a mortar based formulation
a Lagrange multiplier $\lambda$ is introduced, representing the negative contact traction. With these
definitions at hand the contact virtual work expression is given by

$$\delta \Pi_c = \int_{\gamma_c^{(1)}} \lambda \cdot [\delta u_c^{(1)}(X) - \delta u_c^{(2)}(Y)] \, d\gamma .$$

\section{DISCRETIZATION}

\subsection{Contact virtual work}

The discretized version of contact virtual work is defined by introducing appropriate shape
function expansions for the contact surface fields and the Lagrange multiplier:

$$u|_{\Gamma_c^{(1)}} \approx u_h|_{\Gamma_c^{(1)}/h} = N^S_i d^S_i , \quad i = 1, \ldots, n^S_c ,$$
$$u|_{\Gamma_c^{(2)}} \approx u_h|_{\Gamma_c^{(2)}/h} = N^M_i d^M_i , \quad i = 1, \ldots, n^M_c ,$$
$$\lambda = \lambda_h = \phi_i z_i , \quad i = 1, \ldots, n^S_c.$$

Herein $N^S_i$ and $N^M_i$ are the shape functions for the displacements on the slave and master contact
surface respectively and $\phi_i$ are dual shape functions which fulfill the so-called biorthogonality
condition

$$\int_{\gamma_c^{(1)}} N^S_i \phi_j \, d\gamma = \delta_{ij} \int_{\gamma_c^{(1)}} N^S_i \, d\gamma$$

with the related shape functions on the slave side where $\delta_{ij}$ is the Kronecker delta. The
associated discrete nodal values are denoted by $d^S_i$, $d^M_i$ and $z_i$. Substituting (2) into (1)
while using a Bubnov-Galerkin scheme leads to the discrete vector of contact forces

$$f_c = B_c^T z$$

where $B^T = (0, -M_M, D_S)^T$ where $D_S$ and $M_M$ are coupling matrices arising from the so-called
mortar integrals. Because of (3) the matrix $D_S$ becomes diagonal which simplifies the necessary
linearization and solution process.

\subsection{Weak inpenetrability condition}

Applying the mortar method leads to an integral form of the geometric contact constraint

$$\int_{\gamma_c^{(1)}} \delta \lambda \nu \cdot g \, d\gamma \geq 0$$

which will be discretized in an analogous way than the contact virtual work.
Due to the biorthogonality condition of the basis functions the linearization of the discretized weak inpenetrability condition leads to a decoupling of the contact constraints. Thus in the discrete setting the classical Karush-Kuhn-Tucker conditions can be written as a set of incremental scalar inequality constraints for each individual contact node on $\gamma_c^{(1)n}$:

$$
[\Delta \tilde{d}_\nu]_i \leq (\tilde{g}_{n+1})_i ; \quad (z_\nu)_i \geq 0 ; \quad [z_\nu ([\Delta \tilde{d}_\nu] - \tilde{g}_{n+1})]_i = 0 \quad (4)
$$

Herein $[\Delta \tilde{d}_\nu]_i$ is the weighted incremental normal jump of a discrete slave node defined with

$$
[\Delta \tilde{d}_\nu]_i := (\nu_{k+1})^T_i D_S[i, i] [\Delta d]_i \quad (no \ sum) \quad (5)
$$

and $(\tilde{g}_{n+1})_i$ is the weighted nodal gap at time step $n + 1$ in the Newton iteration $k$.

**4 ACTIVE SET STRATEGY**

In order to take the discrete contact boundary conditions into account a so-called primal-dual active set strategy$^2$ is used. Therefore the set of slave nodes is divided into an active and inactive subset. Consequently the scalar inequality conditions can be transformed into appropriate equality constraints for the two subsets. Then these equality constraints are taken into account via prescribed incremental boundary conditions. To do so a local basis transformation$^3$ of the intrinsic system of equations has to be performed such that the incremental jump of the slave nodes appear as primal unknowns. Because of $D_S$ being a diagonal matrix a static condensation can be carried out which leads to a reduced system of equations to be solved solely for the unknown incremental nodal displacements.

**5 EXAMPLES**

**5.1 Contact patch test**

![Contact patch test figure](image)

Figure 2: Contact patch test: problem definition (left) - deformed and undeformed meshes (right)

A contact formulation should be able to exactly transmit a spatially constant stress field from one body to another. If this is satisfied for an arbitrary, generally non-conforming discretization
along the contact interface, the specific contact formulation passes the contact patch test. In Figure 2 the problem definition as well as the deformed and undeformed meshes of the analyzed example is shown. The patch test is passed exactly.

5.2 Dynamic ball

For the analysis of dynamic contact problems, the proposed contact description is combined with the implicit Generalized Energy-Momentum Method\(^4\). Additionally the Velocity-Update Method\(^5\) is applied to ensure an energy conserving algorithm. To show its performance, an elastic ball is hit between two rigid planes. Although the ball is contacting several times, the proposed solution framework guarantees the exact conservation of the total energy (Figure 3).

![Diagram](image)

Figure 3: Ball hits rigid planes: motion of deformation (left) - energy [J] vs time [s] (right)

6 CONCLUSIONS

A mortar based frictionless contact formulation is presented having two main characteristics:

1. Due to the biorthogonality condition (3) of the basis functions the Lagrange multiplier can be locally eliminated. Thus the size of the system of equations to be solved remains constant during the whole calculation.

2. The formulation does not require any user defined parameter to handle the non-linearity of the contact constraints.

REFERENCES


